

RANKS OF MAHARAM ALGEBRAS

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ABSTRACT. Solving a well-known problem of Maharam, Talagrand [17] constructed an exhaustive non uniformly exhaustive submeasure, thus also providing the first example of a Maharam algebra that is not a measure algebra. To each exhaustive submeasure one can canonically assign a certain countable ordinal, its exhaustivity rank. In this paper, we use carefully constructed Schreier families and norms derived from them to provide examples of exhaustive submeasures of arbitrary high exhaustivity rank. This gives rise to uncountably many non isomorphic separable atomless Maharam algebras.

1. INTRODUCTION

We say that a complete Boolean \mathcal{B} algebra is a *measure algebra* if it admits a strictly positive σ -additive probability measure. Recall that a *submeasure* on Boolean algebra \mathcal{B} is a function $\nu : \mathcal{B} \rightarrow [0, +\infty]$ such that

- (1) $\nu(\mathbf{0}) = 0$,
- (2) If $x \leq y$ then $\nu(x) \leq \nu(y)$,
- (3) $\nu(x \vee y) \leq \nu(x) + \nu(y)$, for all $x, y \in \mathcal{B}$;

We say that ν is *positive* if $\nu(a) > 0$, for every $a \in \mathcal{B} \setminus \{\mathbf{0}\}$. If \mathcal{B} is complete the role of σ -additivity is played by the following continuity condition.

- (4) $\nu(x_n) \rightarrow \nu(\inf_n x_n)$, whenever $\{x_n\}_n$ is a decreasing sequence.

A submeasure ν satisfying (4) is called *continuous*. If a complete Boolean algebra \mathcal{B} carries a positive continuous submeasure then we call it a *Maharam algebra*.

In an attempt to find an algebraic characterization of measure algebras Von Neumann asked in 1937 if every ccc weakly distributive complete Boolean algebra is a measure algebra (see [12]). Working on Von Neumann's problem Maharam [11] formulated the notion of a continuous submeasure and found an algebraic characterization for a complete Boolean algebra to carry one. Maharam also showed that every Maharam algebra is weakly distributive and satisfies the ccc. Therefore Von Neumann's original question was naturally decomposed into two questions.

Question 1. *Is every Maharam algebra a measure algebra?*

Question 2. *Is every ccc weakly distributive complete Boolean algebra a Maharam algebra?*

In this paper we will not discuss Question 2, instead we refer the interested reader to [18]. Over the years a significant amount of work has been done on Question 1, which was known to be equivalent to the famous Control Measure Problem, i.e. the question

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whether every countably additive vector valued measure μ defined on a σ -algebra of sets and taking values in an F -space, i.e. a completely metrizable topological vector space, admits a *control measure*, i.e. a countable additive scalar measure λ having the same null sets as μ . For instance, Kalton and Roberts [9] showed that a submeasure μ defined on a (not necessarily complete) Boolean algebra \mathcal{B} is equivalent to a measure if and only if it is uniformly exhaustive. Recall that a submeasure μ on a Boolean algebra \mathcal{B} is called *exhaustive* if for every sequence $\{a_n\}_n$ of disjoint elements of \mathcal{B} we have $\lim_n \mu(a_n) = 0$, μ is called *uniformly exhaustive* if for every $\epsilon > 0$ there is an integer n such that there is no sequence of n pairwise disjoint elements of \mathcal{B} of μ -submeasure $\geq \epsilon$. Clearly, every continuous submeasure on a complete Boolean algebra is exhaustive. If μ is a positive submeasure on a Boolean algebra \mathcal{B} one can define a metric d on \mathcal{B} by setting $d(a, b) = \mu(a \Delta b)$. If μ is exhaustive then the metric completion $\bar{\mathcal{B}}$ of \mathcal{B} equipped with the natural boolean algebraic structure is a complete Boolean algebra and μ has a unique extension $\bar{\mu}$ to a continuous submeasure on $\bar{\mathcal{B}}$. Thus $\bar{\mathcal{B}}$ is a Maharam algebra. It follows that Question 1 is equivalent to the question whether every exhaustive submeasure on a Boolean algebra \mathcal{B} is uniformly exhaustive. In 2005 Talagrand [17] produced a remarkable example of an exhaustive submeasure which is not uniformly exhaustive. As a consequence he obtained the following result.

Theorem 1.1 ([17]). *There is a Maharam algebra which is not a measure algebra.*

Now we know that there are Maharam algebras that are not measure algebras, but we do not know much about their structure. Fremlin (see [5]) suggested using the exhaustivity rank as a tool for classifying Maharam algebras.

Suppose that \mathcal{B} is a Boolean algebra and ν an exhaustive submeasure on \mathcal{B} . For $\epsilon > 0$, let $\mathcal{D}_\epsilon(\nu)$ be the set of all finite pairwise disjoint subsets F of \mathcal{B} such that $\nu(a) \geq \epsilon$, for all $a \in F$. Since ν is exhaustive it follows that $(\mathcal{D}_\epsilon(\nu), \supset)$ is well-founded. Let $\text{rk}_\epsilon(\nu)$ be the rank of this ordering. More precisely, for each $F \in \mathcal{D}_\epsilon(\nu)$, we define the $\text{rk}_\epsilon(\nu, F)$ by letting:

$$\text{rk}_\epsilon(\nu, F) = \sup\{\text{rk}_\epsilon(\nu, G) + 1 : G \in \mathcal{D}_\epsilon(\nu) \text{ and } G \supsetneq F\}$$

We then let $\text{rk}_\epsilon(\nu) = \text{rk}_\epsilon(\nu, \emptyset)$. Finally, we let $\text{rk}(\nu) = \sup\{\text{rk}_\epsilon(\nu) : \epsilon > 0\}$. Since any two Maharam submeasures on a Maharam algebra \mathcal{B} are absolutely continuous with respect to each other they have the same exhaustivity rank, hence this rank is an invariant of \mathcal{B} and we denote it by $\text{rk}(\mathcal{B})$. Fremlin [4] proved that if \mathcal{B} is a Maharam algebra, but not a measure algebra then $\text{rk}(\mathcal{B}) \geq \omega^\omega$. He also showed that $\text{rk}(\mathcal{T}) \leq \omega^{\omega^2}$ for the Maharam algebra \mathcal{T} constructed by Talagrand [17]. Generalizing Fremlin's question ([5] 539Z) we consider the following.

Question 3. *Are there Maharam algebras of arbitrary high countable exhaustivity rank?*

We give a positive answer to this question. We define the notion of an admissible norm and we generalize Talagrand's construction by replacing the cardinalities of the relevant sets by their norms. By varying this norm we obtain examples of submeasures of arbitrary high exhaustivity ranks.

The paper is organized as follows. In §2 we define admissible norms and show how to produce examples of such norms using Schreier families. We also prove some easy technical facts about these norms that will be needed in the main construction. In §3 we

describe our generalization of Talagrand's construction based on any admissible norm. We also give a lower bound on the exhaustivity ranks of the submeasures built on the Schreier norms. In §4 we prove that the submeasures constructed in §3 are exhaustive and derive some corollaries. In §5 we provide upper bounds on the exhaustivity ranks of our submeasures. Our presentation is completely self-contained, however a good understanding of [2], [13], and [17] would clearly be useful when reading the current paper.

2. ADMISSIBLE FAMILIES AND NORMS

We will be interested in functions on finite sets of integers that have certain features of the cardinality function.

Definition 2.1. Suppose A and B are finite subsets of \mathbb{N} . We write $A \leq_s B$ if, letting $A = \{a_0, \dots, a_{n-1}\}$ and $B = \{b_0, \dots, b_{m-1}\}$ be the increasing enumerations of A and B , we have that $n = m$ and $a_i \leq b_i$, for all $i < n$.

Definition 2.2. A norm is a function $\|\cdot\| : [\mathbb{N}]^{<\omega} \rightarrow \mathbb{N}$ such that:

- (1) $\|\emptyset\| = 0$ and $\|\{n\}\| = 1$, for all $n \in \mathbb{N}$,
- (2) if $A \subseteq B$ then $\|A\| \leq \|B\|$,
- (3) $\|A \cup B\| \leq \|A\| + \|B\|$, for every $A, B \in [\mathbb{N}]^{<\omega}$.

We say that a norm $\|\cdot\|$ is:

- (4) unbounded if $\lim_{n \rightarrow \infty} \|A \cap n\| = +\infty$, for every infinite $A \subseteq \mathbb{N}$,
- (5) spreading if $\|A\| \leq \|B\|$, for every $A, B \in [\mathbb{N}]^{<\omega}$ such that $A \leq_s B$.

A norm that is both unbounded and spreading will be called admissible.

We now describe a canonical way to generate admissible norms on $[\mathbb{N}]^{<\omega}$.

Definition 2.3. Let \mathcal{S} be a family of finite subsets of \mathbb{N} . We say that \mathcal{S} is:

- (1) hereditary if it is closed under taking subsets.
- (2) spreading if $A \in \mathcal{S}$ and $A \leq_s B$ implies $B \in \mathcal{S}$.
- (3) compact if it is a compact subset of $2^{\mathbb{N}}$ with the product topology, where we identify a subset of \mathbb{N} with its characteristic function.

Finally, we say that \mathcal{S} is admissible if it is compact, hereditary, spreading and contains all singletons.

Suppose \mathcal{S} is an admissible family of finite subsets of \mathbb{N} . We can define a norm $\|\cdot\|_{\mathcal{S}}$ by letting $\|A\|_{\mathcal{S}}$ be the least number of members of \mathcal{S} needed to cover A . It is straightforward to check that $\|\cdot\|_{\mathcal{S}}$ is an admissible norm. Conversely, if $\|\cdot\|$ is an admissible norm we can let $\mathcal{S} = \{F \in [\mathbb{N}]^{<\omega} : \|F\| \leq 1\}$. Then \mathcal{S} is an admissible family and $\|\cdot\| = \|\cdot\|_{\mathcal{S}}$. We can assign a rank to each admissible family \mathcal{S} . We do this using the language of games.

Definition 2.4. Let \mathcal{S} be an admissible family and α a countable ordinal. The game $\mathcal{G}_{\alpha}(\mathcal{S})$ is played between two players I and II as follows.

$$\begin{array}{cccccc} \text{I :} & \alpha_0 & \alpha_1 & \cdots & \alpha_k & \cdots \\ \hline \text{II :} & n_0 & n_1 & \cdots & n_k & \cdots \end{array}$$

Player I is required to play a decreasing sequence of ordinals $\leq \alpha$ and Player II is required to play an increasing sequence of integers such that $\{n_0, \dots, n_k\} \in \mathcal{S}$, for all k . The first player who cannot play loses.

Since Player I plays a decreasing sequence of ordinals, the game must end after finitely many stages. Therefore, by the Gale-Stewart theorem [6] one of the players has a winning strategy. Since every infinite subset of \mathbb{N} has an initial segment which is not in \mathcal{S} , Player II cannot have a winning strategy in $\mathcal{G}_\alpha(\mathcal{S})$, for all $\alpha < \omega_1$. We let $\rho(\mathcal{S})$ be the least α such that Player I has a winning strategy in $\mathcal{G}_\alpha(\mathcal{S})$. Let $T_\mathcal{S}$ be the set of strictly increasing sequences of integers whose range is in \mathcal{S} . We order $T_\mathcal{S}$ by reverse extension, i.e. $s < t$ iff t is a proper initial segment of s . Then $T_\mathcal{S}$ is well-founded and $\rho(\mathcal{S})$ is simply the well-founded rank of $T_\mathcal{S}$.

Given families \mathcal{S} and \mathcal{T} of subsets of \mathbb{N} let $\mathcal{S} \oplus \mathcal{T} = \{S \cup T : S \in \mathcal{S} \text{ and } T \in \mathcal{T}\}$. It is easy to see that, if \mathcal{S} and \mathcal{T} are admissible, then so is $\mathcal{S} \oplus \mathcal{T}$.

Lemma 2.5. *Suppose \mathcal{S} and \mathcal{T} are admissible families. Let $\rho(\mathcal{S}) = \alpha$ and $\rho(\mathcal{T}) = \beta$. Then $\rho(\mathcal{S} \oplus \mathcal{T}) \leq (\alpha + 1)(\beta + 1) - 1$.*

Proof. Let us fix winning strategies σ and τ for Player I in $\mathcal{G}_\alpha(\mathcal{S})$ and $\mathcal{G}_\beta(\mathcal{T})$. Let $\gamma = (\alpha + 1)(\beta + 1) - 1$. We need to define a winning strategy for Player I in $\mathcal{G}_\gamma(\mathcal{S} \oplus \mathcal{T})$. Note that the lexicographic ordering $<_{\text{lex}}$ on $(\beta + 1) \times (\alpha + 1)$ has order type $(\alpha + 1)(\beta + 1)$. So, instead of playing ordinals $\leq \gamma$ Player I will play pairs of ordinals in $(\beta + 1) \times (\alpha + 1)$ decreasing under $<_{\text{lex}}$. Player I starts by playing according to σ , but at any give stage instead of playing the ordinal ξ given by σ he plays (β, ξ) . Player II plays an increasing sequence of integers $\{n_0, n_1, \dots\}$. Since σ is a winning strategy for Player I in $\mathcal{G}_\alpha(\mathcal{S})$ there must be a stage k_0 such that $\{n_0, \dots, n_{k_0}\} \notin \mathcal{S}$. At that moment Player I switches to playing $\mathcal{G}_\beta(\mathcal{T})$ and considers that Player II has played n_{k_0} as the first move in this game. Suppose τ replies by playing some $\beta_1 < \beta$. Player I then starts a new run of $\mathcal{G}_\alpha(\mathcal{S})$ in which he plays pairs of the form (β_1, ξ) , for $\xi \leq \alpha$. Since σ is a winning strategy in this game, there must be a first stage k_1 such that $\{n_{k_0+1}, \dots, n_{k_1}\} \notin \mathcal{S}$. Player I then considers that Player II has made another move in $\mathcal{G}_\beta(\mathcal{T})$ by playing n_{k_1} . Let $\beta_2 < \beta_1$ be the response of τ . Player I then starts yet another run of $\mathcal{G}_\alpha(\mathcal{S})$ in which he plays pairs of the form (β_2, ξ) , for $\xi \leq \alpha$. Continuing in this way, we obtain increasing blocks of integers B_0, B_1, \dots . Each block B_i is of the form $\{n_{k_{i-1}+1}, \dots, n_{k_i}\}$. Here, we set by convention $k_{-1} = -1$. We have that $B_i \notin \mathcal{S}$, for each i . Since τ is a winning strategy for Player I in $\mathcal{G}_\beta(\mathcal{T})$, by the time Player I reaches $(0, 0)$, Player II has played l blocks B_0, \dots, B_{l-1} such that $R = \{n_{k_0}, \dots, n_{k_{l-1}}\} \notin \mathcal{T}$. We claim that $\bigcup_{i < l} B_i \notin \mathcal{S} \oplus \mathcal{T}$. Indeed, suppose it could be written as $S \cup T$, for some $S \in \mathcal{S}$ and $T \in \mathcal{T}$. Since $B_i \notin \mathcal{S}$, there must be an element $m_i \in B_i \setminus S$, for all $i < l$. Let $P = \{m_0, \dots, m_{l-1}\}$. Then we must have that $P \subseteq T$. Since \mathcal{T} is hereditary, we would have that $P \in \mathcal{T}$, as well. Now, note that $P \leq_s R$. Since \mathcal{T} is also spreading, we would get that $R \in \mathcal{T}$, a contradiction. □

Corollary 2.6. *Suppose \mathcal{S} is an admissible family and let $\alpha = \rho(\mathcal{S})$. Let $\|\cdot\|_\mathcal{S}$ be the associated norm. Suppose $n > 0$ is an integer and let $\mathcal{S}^n = \{F \in [\mathbb{N}]^{<\omega} : \|F\|_\mathcal{S} \leq n\}$. Then $\rho(\mathcal{S}^n) \leq (\alpha + 1)^n - 1$. □*

We now define a version of the Schreier families initially introduced in [15]. These families have played an important role in the theory of Banach spaces, see for instance [1] or [7]. For applications of Schreier families in combinatorics, see, for instance, [3]. Since we need our families to be spreading we have to take some care in their definition. It will be convenient to use the following lemma of Galvin, see [8] or [14] for a proof.

Lemma 2.7. *There is a sequence $(<_n)_n$ of tree orderings on ω_1 such that:*

- (1) *if $n < m$ and $\xi <_n \eta$ then $\xi <_m \eta$,*
- (2) *$(\omega_1, <_n)$ has finite height, for all n ,*
- (3) *$<\upharpoonright \omega_1 = \bigcup_n <_n$.*

□

We fix a Galvin decomposition $(<_n)_n$ such that $0 <_n \xi$, for all $0 < \xi < \omega_1$ and all n .

Definition 2.8 (Schreier families). *Suppose $0 < \alpha < \omega_1$. We define the family \mathcal{S}_α as the collection of all $F \in [\mathbb{N}]^{<\omega}$ such that, if $F = \{n_0, \dots, n_{k-1}\}$ is the increasing enumeration, there is a sequence $(\alpha_i)_{i \leq k}$ of ordinals such that $\alpha_0 = \alpha$ and $\alpha_{i+1} <_{n_i} \alpha_i$, for all $i < k$.*

Lemma 2.9. *The family \mathcal{S}_α is admissible, for all countable ordinals $\alpha > 0$.*

Proof. Let us first observe that if $F \in \mathcal{S}_\alpha$ then there is a canonical sequence of ordinals witnessing it. Namely, suppose $F = \{n_0, \dots, n_{k-1}\}$ is the increasing enumeration. We define the sequence $(\alpha_i)_{i \leq k}$ by induction as follows. Let $\alpha_0 = \alpha$. Suppose α_i has been defined. Since $(\omega_1, <_{n_i})$ is a tree of finite height, the set of $<_{n_i}$ -predecessors of α_i is finite and totally ordered. If it is non-empty, we let α_{i+1} be the largest $<_{n_i}$ -predecessor of α_i . It is straightforward to check, by using (1) of Definition 2.7 and the fact that $F \in \mathcal{S}_\alpha$, that we can continue the construction up to k . Notice that, also by (1) of Definition 2.7, the family \mathcal{S}_α is spreading and hereditary, for all α . To see that \mathcal{S}_α is compact, suppose A is an infinite subset of \mathbb{N} such that $A \cap n \in \mathcal{S}_\alpha$, for all n . Let $\{n_0, n_1, \dots\}$ be the increasing enumeration of A . Then, as before, we could construct a sequence $(\alpha_k)_k$ of ordinals such that $\alpha_0 = \alpha$ and $\alpha_{k+1} <_{n_k} \alpha_k$, for all k . Then $(\alpha_k)_k$ would be an infinite decreasing sequence of ordinals, a contradiction. Finally, if $\alpha > 0$, since we assumed that $\alpha >_n 0$, for all n , it follows that \mathcal{S}_α contains all singletons. Therefore, \mathcal{S}_α is an admissible family. Let us also note that if $\alpha <_n \beta$, $n < F$ and $F \in \mathcal{S}_\alpha$ then $F \cup \{n\} \in \mathcal{S}_\beta$. □

Lemma 2.10. *$\rho(\mathcal{S}_\alpha) = \alpha$, for all countable ordinals $\alpha > 0$.*

Proof. Suppose $\alpha^* < \alpha$. We describe a winning strategy for Player II in $\mathcal{G}_{\alpha^*}(\mathcal{S}_\alpha)$. We may assume that Player I starts by playing $\alpha_0 = \alpha^*$. Player II lets n_0 be the least integer such that $\alpha_0 <_{n_0} \alpha$. At stage k , suppose Player I plays some $\alpha_k < \alpha_{k-1}$. Then Player II lets n_k be the least integer bigger than n_{k-1} such that $\alpha_k <_{n_k} \alpha_{k-1}$. Since $\alpha >_{n_0} \alpha_0 > \dots >_{n_k} \alpha_k$, it follows that $\{n_0, \dots, n_k\} \in \mathcal{S}_\alpha$. This means that Player II can keep playing as long as Player I keeps producing a decreasing sequence of ordinals. Therefore, Player II wins by playing in this way.

Now we describe the winning strategy for Player I in $\mathcal{G}_\alpha(\mathcal{S}_\alpha)$. He starts by playing $\alpha_0 = \alpha$. Suppose Player II responds by playing some n_0 . Since $\{n_0\} \in \mathcal{S}_\alpha$ then α_0 is not a minimal element in $<_{n_0}$. Let α_1 be the largest $<_{n_0}$ -predecessor of α_0 . Player I then plays α_1 . Suppose we are at some stage k and Player II has played n_{k-1} in the previous stage. Since $\{n_0, \dots, n_{k-1}\} \in \mathcal{S}_\alpha$ it follows that α_{k-1} is not a minimal element

in $<_{n_{k-1}}$. Then Player I plays as α_k the largest $<_{n_{k-1}}$ -predecessor of α_{k-1} . Since $(\alpha_k)_k$ is a decreasing sequence of ordinals, the game must stop at some stage, i.e. at some k Player II cannot find $n_k > n_{k-1}$ such that $\{n_0, \dots, n_k\} \in \mathcal{S}_\alpha$. Therefore, Player I wins by following this strategy. \square

Definition 2.11. We shall write $\|\cdot\|_\alpha$ for the norm derived from the family \mathcal{S}_α , for $\alpha < \omega_1$.

We now turn to a different game that will be used to analyze the exhaustivity ranks of our submeasures.

Definition 2.12. Suppose \mathcal{P} is a poset and $\mathcal{F} \subseteq \mathcal{P}$. For an ordinal α the game $\mathcal{H}_\alpha(\mathcal{F})$ is played between players I and II as follows.

$$\begin{array}{ccccccc} \text{I :} & \alpha_0 & \alpha_1 & \cdots & \alpha_k & \cdots \\ \hline \text{II :} & p_0 & p_1 & \cdots & p_k & \cdots \end{array}$$

Player I is required to plays decreasing sequence of ordinals $\leq \alpha$, while Player II plays pairwise incompatible members of \mathcal{F} . The first player who cannot play loses.

Clearly, if there is an infinite pairwise incompatible sequence of elements of \mathcal{F} then Player II has a winning strategy in $\mathcal{H}_\alpha(\mathcal{F})$, for any α . He simply plays the members of that sequence regardless of what Player I plays. If there is no such sequence of members of \mathcal{F} then there is an ordinal α such that Player I has a winning strategy. Let $\delta(\mathcal{F})$ be the least such α . In other words, if $\mathcal{D}(\mathcal{F})$ is the family of pairwise incompatible finite subsets of \mathcal{F} then $\delta(\mathcal{F})$ is equal to $\text{rk}(\mathcal{D}(\mathcal{F}))$, i.e. the rank of $\mathcal{D}(\mathcal{F})$ under reverse inclusion.

In the next lemma and in §5 we will use the natural sum of ordinals, see [16]. Recall that for ordinals α and β , the *natural sum* of α and β , denoted by $\alpha \oplus \beta$ is defined by simultaneous induction on α and β as the smallest ordinal greater than $\alpha \oplus \gamma$, for all $\gamma < \beta$, and $\gamma \oplus \beta$, for all $\gamma < \alpha$. Another way to define the natural sum of two ordinals α and β is to use the Cantor normal form: one can find a sequence of ordinals $\gamma_0 > \dots \gamma_{n-1}$ and two sequences (k_0, \dots, k_{n-1}) and (j_0, \dots, j_{n-1}) of natural numbers (including zero, but satisfying $k_i + j_i > 0$, for all i) such that $\alpha = \omega^{\gamma_0} \cdot k_0 + \dots + \omega^{\gamma_{n-1}} \cdot k_{n-1}$ and $\beta = \omega^{\gamma_0} \cdot j_0 + \dots + \omega^{\gamma_{n-1}} \cdot j_{n-1}$ and defines

$$\alpha \oplus \beta = \omega^{\gamma_0} \cdot (k_0 + j_0) + \dots + \omega^{\gamma_{n-1}} \cdot (k_{n-1} + j_{n-1}).$$

What is important for us is that the natural sum is associative and commutative. It is always greater or equal to the usual sum, but it may be strictly greater.

Definition 2.13. Let \mathcal{P} be the poset of all partial functions u such that $\text{dom}(u) \in [\mathbb{N}]^{<\omega}$ and $u(k) < 2^k$, for all $k \in \text{dom}(u)$, ordered under reverse inclusion. For $0 < \alpha < \omega_1$, let \mathcal{P}_α be the set of all $u \in \mathcal{P}$ with $\|\text{dom}(u)\|_\alpha \leq 1$.

Lemma 2.14. Suppose $0 < \alpha < \omega_1$. Then $\delta(\mathcal{P}_\alpha) = \omega^\alpha$.

Proof. We will give a proof by induction. First note that, since \mathcal{S}_β is spreading, for all β , if Player II has a winning strategy in $\mathcal{H}_\gamma(\mathcal{P}_\beta)$ for some γ , then for every integer n , Player II has a winning strategy in the same game in which he plays partial functions

$u \in \mathcal{P}_\beta$ with $\min(\text{dom}(u)) > n$. Now, suppose α is a countable ordinal and the statement is true for all $\beta < \alpha$.

Suppose $\xi < \omega^\alpha$. We describe informally a winning strategy for Player II in $\mathcal{H}_\xi(\mathcal{P}_\alpha)$. We may assume that Player I's starts by playing $\xi_0 = \xi$. We first find $\beta < \alpha$ and an integer n_0 such that $\xi_0 = \omega^\beta \cdot n_0 + \eta_0$, for some $\eta_0 < \omega^\beta$. We can then find an integer m such that $2^m > n_0$ and $\beta <_m \alpha$. Note that if $F \in \mathcal{S}_\beta$ and $m < \min(F)$ then $\{m\} \cup F \in \mathcal{S}_\alpha$. Fix a winning strategy τ_0 for Player II in $\mathcal{H}_{\eta_0}(\mathcal{P}_\beta)$ in which he plays only partial functions $u \in \mathcal{P}_\beta$ with $\min(\text{dom}(u)) > m$. Now, let u_0 be the response of τ_0 if Player I plays η_0 in the game $\mathcal{H}_{\eta_0}(\mathcal{P}_\beta)$. Player II then plays $v_0 = \{(m, n_0)\} \cup u_0$. Note that if $F_0 = \text{dom}(u_0)$ then $F_0 \in \mathcal{S}_\beta$ and hence $\{m\} \cup F_0 \in \mathcal{S}_\alpha$. In other words $v_0 \in \mathcal{F}_\alpha$. As long as Player I plays ordinals ξ_i of the form $\omega^\beta \cdot n_0 + \eta_i$, for some η_i , Player II simulates the run of the game $\mathcal{H}_{\eta_0}(\mathcal{P}_\beta)$ in which Player I plays the η_i . At stage i , if u_i is the response of τ_0 in that game he plays $v_i = \{(m, n_0)\} \cup u_i$ in the current game. Suppose that at some stage i Player I plays an ordinal ξ_i of the form $\omega^\beta \cdot n_1 + \eta_i$ for some $n_1 < n_0$ and $\eta_i < \omega^\beta$. Fix a winning strategy τ_1 for Player II in $\mathcal{H}_{\eta_i}(\mathcal{P}_\beta)$ in which he plays only partial functions $u \in \mathcal{P}_\beta$ with $\min(\text{dom}(u)) > m$. Let u_i be the reply of τ_1 if Player I starts by playing η_i in $\mathcal{H}_{\eta_i}(\mathcal{P}_\beta)$. Then Player II plays $v_i = \{(m, n_1)\} \cup u_i$ in the current game. As before, we have that $v_i \in \mathcal{P}_\alpha$. Proceeding in this way, Player II plays pairwise incompatible members of \mathcal{P}_α as long as the game last. Thus Player II has a winning strategy in $\mathcal{H}_\xi(\mathcal{P}_\alpha)$, as desired.

We now show that Player I has a winning strategy in $\mathcal{H}_{\omega^\alpha}(\mathcal{P}_\alpha)$. Of course, Player I starts by playing ω^α . Suppose Player II responds by playing some $u_0 \in \mathcal{P}_\alpha$. Fix an integer m such that $\text{dom}(u_0) \subseteq m$ and let β be the immediate $<_m$ -predecessor of α . First note that if $u \in \mathcal{P}_\alpha$ is incompatible with u_0 then $\text{dom}(u) \cap m \neq \emptyset$ and $\text{dom}(u) \setminus m \in \mathcal{P}_\beta$. Let \mathcal{D} be the set of all $s \in \mathcal{P}_\alpha$ which are nonempty and such that $\text{dom}(s) \subseteq m$. Note that \mathcal{D} is finite. Let t be the cardinality of \mathcal{D} and let $\{s_0, \dots, s_{t-1}\}$ be an enumeration of \mathcal{D} . By the inductive assumption, there is a winning strategy, say τ , for Player I in $\mathcal{H}_{\omega^\beta}(\mathcal{P}_\beta)$. On the side, Player I starts t runs of $\mathcal{H}_{\omega^\beta}(\mathcal{P}_\beta)$ simultaneously in which he simulates the moves of Player II and uses the responses of τ in order to produce a move in $\mathcal{H}_{\omega^\alpha}(\mathcal{P}_\alpha)$. We may assume that the first move of τ is ω^β . Player I then plays $\omega^\beta \cdot t$ in $\mathcal{H}_{\omega^\alpha}(\mathcal{P}_\alpha)$. At stage i suppose Player II plays $u_i \in \mathcal{P}_\alpha$ that is incompatible with the u_j , for $j < i$. In particular, u_i is incompatible with u_0 and hence $\text{dom}(u) \cap m \neq \emptyset$. Let $\xi_{k,i-1}$ be the latest ordinal played by τ in the k -th run of $\mathcal{H}_{\omega^\beta}(\mathcal{P}_\beta)$. Let $r < t$ be such that $u_i \upharpoonright m = s_r$. Player I then considers the r -th run of $\mathcal{H}_{\omega^\beta}(\mathcal{P}_\beta)$ and simulates a move of Player II in that game by playing $u_i \upharpoonright [m, \omega)$. Note that $u_i \upharpoonright [m, \omega) \in \mathcal{P}_\beta$ and is incompatible with $u_j \upharpoonright [m, \omega)$, for all $j < i$ such that $u_j \upharpoonright m = s_r$. Thus, $u_i \upharpoonright [m, \omega)$ is a legitimate move by Player II in that position of $\mathcal{H}_{\omega^\beta}(\mathcal{P}_\beta)$. Let $\xi_{r,i}$ be the response of τ . For all $k \neq r$, Player I considers that no move is made in the k -th copy of $\mathcal{H}_{\omega^\beta}(\mathcal{P}_\beta)$ and sets $\xi_{k,i} = \xi_{k,i-1}$. Finally, in $\mathcal{H}_{\omega^\alpha}(\mathcal{P}_\alpha)$, Player I plays

$$\xi_i = \xi_{0,i} \oplus \xi_{1,i} \oplus \dots \oplus \xi_{t-1,i}.$$

Since $\xi_{r,i} < \xi_{r,i-1}$ and $\xi_{k,i} = \xi_{k,i-1}$, for all $k \neq r$, it follows that $\xi_i < \xi_{i-1}$. Since τ is a winning strategy for Player I in $\mathcal{H}_{\omega^\beta}(\mathcal{P}_\beta)$, it follows that Player I can continue playing in this way as long as Player II plays pairwise incompatible members of \mathcal{P}_α . Hence, this is a winning strategy for Player I in $\mathcal{H}_{\omega^\alpha}(\mathcal{P}_\alpha)$, as required. \square

We shall need a version of the following lemma due to Roberts [13].

Lemma 2.15 (Roberts' Selection Lemma). *Let $\|\cdot\|$ be an admissible norm. Suppose s, t are integers and I_l is a finite subset of \mathbb{N} with $\|I_l\| \geq st$, for all $l < s$. Then there is a permutation π of $\{0, \dots, s-1\}$ and sets $J_i \subseteq I_{\pi(i)}$, for all $i < s$, such that $J_1 < J_2 < \dots < J_s$ and $\|J_i\| = t$, for all $i < s$.*

Proof. We essentially repeat the original argument. We define integers k_i and $\pi(i)$, with $\pi(i) < s$, and sets $J_i \subseteq I_{\pi(i)}$ by induction on $i < s$. To begin, by (1) of Definition 2.2, we can find the least integer k_0 such that $\|I_l \cap k_0\| = t$, for some $l < s$. We let $\pi(0)$ be the least such l and let $J_0 = I_{\pi(0)} \cap k_0$. Note that, again by (1) of Definition 2.2, $\|I_l \setminus k_0\| \geq (s-1)t$, for all $l \neq \pi(0)$. Having defined k_j and $\pi(j)$, for all $j < i$, let k_i be the least integer such that $\|I_l \cap [k_{i-1}, k_i]\| \geq t$, for some $l \neq \pi(0), \dots, \pi(i-1)$. Let $\pi(i)$ be the least such l and let $J_i = I_{\pi(i)} \cap [k_{i-1}, k_i]$. We can clearly continue the construction for all $i < s$. \square

We shall also need the following simple lemma which is the main reason why we require our admissible norms to be spreading.

Lemma 2.16. *Let $\|\cdot\|$ be an admissible norm. Suppose $C, D \subseteq \mathbb{N}$ are such that $\|C\| \geq 3$ and $\|D\| = 1$. Then there are consecutive elements c, d of C such that $[c, d) \cap D = \emptyset$.*

Proof. Let t be the cardinality of C and let $\{c_i : i < t\}$ be the increasing enumeration of C . Suppose $D \cap [c_i, c_{i+1}) \neq \emptyset$, for all $i < t-1$, and pick $e_i \in D \cap [c_i, c_{i+1})$, for all $i < t-1$. Let $C' = C \setminus \{c_0\}$ and $E = \{e_i : i < t-1\}$. Since $E \subseteq D$ and $\|D\| = 1$ it follows that $\|E\| \leq 1$. Since $E \leq_s C'$ and $\|\cdot\|$ is spreading we also get that $\|C'\| \leq 1$. Now, $\|\{c_0\}\| = 1$, hence, by subadditivity of $\|\cdot\|$ we get that $\|C\| \leq 2$, a contradiction. \square

3. TALAGRAND'S CONSTRUCTION REVISITED

In this section we associate to each admissible norm $\|\cdot\|$ an exhaustive submeasure on a countable atomless Boolean algebra. The construction generalizes the one of Talagrand [17], which itself builds on previous work of Roberts [13] and Farah [2]. In our case special care has to be taken in order to take into account the fact that $\|\cdot\|$ is only subadditive rather than additive. We start by describing the topological space and Boolean algebra that we will work with.

Let $T = \prod_n 2^n$. For $n \in \mathbb{N}$, let \mathcal{B}_n denote the algebra of subsets of T that depend only on the coordinates $< n$. Then $\mathcal{B} = \bigcup_n \mathcal{B}_n$ is the algebra of clopen subsets of T . We denote by \mathcal{A}_n the set of atoms of \mathcal{B}_n and call them the atoms of rank n . For $X \subseteq T$ we will write

$$[X]_n = \bigcap \{B \in \mathcal{B}_n : X \subseteq B\} = \bigcup \{A \in \mathcal{A}_n : A \cap X \neq \emptyset\}$$

to describe the smallest clopen set in \mathcal{B}_n containing X . We also write $\text{int}_n(X)$ for the largest clopen set in \mathcal{B}_n contained in X , i.e.

$$\text{int}_n(X) = \bigcup \{A \in \mathcal{A}_n : A \subseteq X\}.$$

Let us recall that \mathcal{P} denotes the collection of all partial functions u such that $\text{dom}(u) \in [\mathbb{N}]^{<\omega}$ and $u(k) < 2^k$, for all $k \in \text{dom}(u)$. If $u \in \mathcal{P}$ we let $N_u = \{x \in T : u \subseteq x\}$. Then \mathcal{A}_n is precisely the set of the N_u , for $u \in \mathcal{P}$ with $\text{dom}(u) = n$.

Fix, for the rest of this and the next section, an admissible norm $\|\cdot\|$. Our goal is to define a positive exhaustive submeasure $\nu : \mathcal{B} \rightarrow [0, +\infty]$ such that $\nu(N_u) \geq 8$, for all $u \in \mathcal{P}$ with $\|\text{dom}(u)\| \leq 1$. If $\|\cdot\|_\alpha$ is the admissible norm derived from the family \mathcal{S}_α from the previous section, by Lemma 2.14 we have that $\text{rk}(\nu) \geq \omega^\alpha$. In the last section we will also give an upper bound on $\text{rk}(\nu)$.

In order to define our submeasures we will use classes \mathcal{F} of marked weighted sets, objects that have three components: the first one is a clopen subset of T , the second one is a finite set of coordinates and the third is a nonnegative real called the *weight* of the marked set.

Definition 3.1. For $\mathcal{F} \subseteq \mathcal{B} \times \mathcal{P}(\mathbb{N}) \times \mathbb{R}^+$,

$$X(\mathcal{F}) = \bigcup_{(X,I,w) \in \mathcal{F}} X, \text{ and}$$

$$w(\mathcal{F}) = \sum_{(X,I,w) \in \mathcal{F}} w.$$

We use the classes of marked weighted sets to define outer submeasures on \mathcal{B} .

Definition 3.2. For $\mathcal{E} \subseteq \mathcal{B} \times \mathcal{P}(\mathbb{N}) \times \mathbb{R}^+$, define $\phi_{\mathcal{E}} : \mathcal{B} \rightarrow \mathbb{R}^+$ by setting

$$\phi_{\mathcal{E}}(X) = \inf\{w(\mathcal{F}) : \mathcal{F} \subseteq \mathcal{E} \text{ is finite and } X \subseteq X(\mathcal{F})\}$$

By convention, we let $\phi_{\mathcal{E}}(\emptyset) = 0$ and $\phi_{\mathcal{E}}(X) = +\infty$, for all $X \in \mathcal{B}$ that is not covered by $X(\mathcal{F})$, for any finite $\mathcal{F} \subseteq \mathcal{E}$.

The following notation will be frequently used. In particular, if $A \in \mathcal{A}_m$, for some m , and $C \subseteq A$, we will use it to define the relative submeasure of C inside A .

Definition 3.3. Let $A \in \mathcal{A}_m$ and let $u \in \mathcal{P}$ be such that $A = N_u$. Define $\pi_A : T \rightarrow A$ by

$$\pi_A(z)(i) = \begin{cases} u(i), & \text{if } i < m, \\ z(i), & \text{otherwise.} \end{cases}$$

We now recall the definition of a thin set relative to a given submeasure. This notion was initially introduced by Farah in [2] who used it to construct examples of ϵ -exhaustive pathological submeasures. It also plays a key role in Talagrand's construction.

Definition 3.4. Suppose $m < n$, $\phi : \mathcal{B} \rightarrow [0, +\infty]$ is a function and $X \in \mathcal{B}$, then X is (m, n, ϕ) -thin if for all $A \in \mathcal{A}_m$ there is $H \in \mathcal{B}_n$ such that $H \subseteq A \setminus X$ and $\phi(\pi_A^{-1}(H)) > 1$. For $I \subseteq \mathbb{N}$, X is (I, ϕ) -thin I , if it is (m, n, ϕ) -thin, for all $m, n \in I$ with $m < n$.

Notice that $\pi_A^{-1}(H)$ is obtained by simply copying H inside all other atoms in \mathcal{A}_m . In all our cases ϕ will be a submeasure and we think of $\phi(\pi_A^{-1}(H))$ as the submeasure of H relative to A . Therefore, saying that X is (m, n, ϕ) -thin simply means that $\text{int}_n(A \setminus X)$ is large, i.e. has submeasure bigger than 1 relative to A , for every $A \in \mathcal{A}_m$.

Before we present the details, let us describe the main ideas of the construction. We shall fix a sequence of positive reals $(a_k)_k$ such that $\sum_k a_k$ converges and a sequence of integers $(M_k)_k$ quickly increasing to $+\infty$. We identify two properties of a submeasure ϕ which together imply that ϕ is exhaustive.

Definition 3.5. Let k be an integer. We say that a submeasure ϕ on \mathcal{B} has the k -thinness property if $\phi(X) \leq 2^{-k}$, for every $X \in \mathcal{B}$ which is (I, ϕ) -thin for some set I with $\|I\| = M_k$.

Note that this notion depends on our chosen norm $\|\cdot\|$. If the norm is not clear from the context we will explicitly specify it. The next definition is more technical, it expresses a form of regularity of a submeasure ϕ . It is motivated by the notion of a *potentially exhaustive* submeasure introduced in [2].

Definition 3.6. Let ϕ be a submeasure on \mathcal{B} . Suppose m is an integer and $E \in \mathcal{B}$ does not depend on coordinates $< m$ and $\phi(E) < 2$. Let $n(E)$ be the least integer n such that $E \in \mathcal{B}_n$. A sequence $\{C_r^m(E) : m < r \leq n(E)\}$ is an m -covering sequence for E if:

- (1) $C_r^m(E) \in \mathcal{B}_r$, for every r such that $m < r \leq n(E)$,
- (2) $\text{int}_j(E) \subseteq \bigcup \{C_r^m(E) : m < r \leq j\}$, for every j such that $m < j \leq n(E)$,
- (3) $\sum_{m < r \leq n(E)} \phi(C_r^m(E)) \leq 4$.

We say that ϕ has the m -covering property if every such E has an m -covering sequence. Finally, we say that ϕ has the covering property if it has the m -covering property, for every m .

It will be fairly easy to show that if ϕ is a submeasure satisfying the covering and thinness properties and such that $\phi(T) \geq 8$ then ϕ is exhaustive. In order to construct such ϕ the natural idea is to define a sequence $(\mathcal{F}_k)_k$ of subsets of $\mathcal{B} \times \mathcal{P}(\mathbb{N}) \times \mathbb{R}^+$ as follows. Start with $\mathcal{F}_0 = \emptyset$. Given \mathcal{F}_k let $\phi_k = \phi_{\mathcal{F}_k}$. Construct \mathcal{F}_{k+1} by adding to \mathcal{F}_k all triples (X, I, w) such that X is (I, ϕ_k) -thin, $\|I\| \leq M_k$ and

$$w \geq 2^{-k} \left(\frac{M_k}{\|I\|} \right)^{a_k}.$$

The reason for this last requirement is to ensure the covering property. Namely, suppose $(X, I, w) \in \mathcal{F}_{k+1}$. In some situations we will need to replace (X, I, w) by another triple $(X', I', w') \in \mathcal{F}_{k+1}$, where X' is a superset of X and depends only on coordinates in some interval $[m, n]$, $I' = I \cap [m, n]$, and w' is not too big relative to w . If $\|I'\|$ is not too much smaller than $\|I\|$, the fact that we have a_k in the exponent will allow us to choose w' which is very close to w . This construction would ensure that ϕ_{k+1} satisfies the k -thinness condition and, since the sequence $(\phi_k)_k$ is decreasing, this condition will remain to hold for the later ϕ_l . The problem with this scenario is that, in order to obtain an exhaustive submeasure, we would have to continue this process for all k , but as explained in [2], the limit submeasure $\lim_k \phi_k$ collapses to 0.

The main new idea [17] is to reverse this process. Namely, for each p we define families $\mathcal{C}_{k,p}$, for $k \leq p$, by backwards induction. We can start with $\mathcal{C}_{p,p} = \emptyset$. Given $\mathcal{C}_{k+1,p}$ we let $\nu_{k+1,p} = \phi_{\mathcal{C}_{k+1,p}}$ and we construct $\mathcal{C}_{k,p}$ by adding to $\mathcal{C}_{k+1,p}$ all triples (X, I, w) satisfying the thinness and the weight conditions relative to $\nu_{k+1,p}$. We have that the $\nu_{k,p}$ decrease as k gets smaller, but we are able to guarantee that $\nu_{0,p}(T) \geq 8$. In this way we will have that $\nu_{k,p}$ satisfies the l -thinness property, for all $k \leq l < p$. Then we pick a non principal ultrafilter \mathcal{U} on \mathbb{N} and let $\nu_k = \lim_{p \rightarrow \mathcal{U}} \nu_{k,p}$, for each k . The covering property and the l -thinness property are preserved by taking the \mathcal{U} -limit of submeasures, so the resulting ν_k will all be exhaustive.

We now turn to the details of the construction. We define the sequences $(a_k)_k$ and $(M_k)_k$ as follows:

$$\begin{aligned} a_k &= \frac{1}{(k+5)^3} \\ M_k &= 2^{2k+12} \cdot 2^{(k+4)(k+5)^3}. \end{aligned}$$

Definition 3.7. Fix an integer $p \in \mathbb{N}$. We define families $\mathcal{C}_{k,p}$ for $k \leq p$, by downwards induction on k . Once we have $\mathcal{C}_{k,p}$ we let $\nu_{k,p} = \phi_{\mathcal{C}_{k,p}}$. We start by letting $\mathcal{C}_{p,p} = \emptyset$. Suppose $k < p$ and $\mathcal{C}_{k+1,p}$ has been defined. We let:

$$\begin{aligned} \mathcal{E}_{k,p} &= \{(E, I, w) : E \in \mathcal{B}, I \subseteq \mathbb{N}, \|I\| \leq M_k, w \geq 2^{-k} \left(\frac{M_k}{\|I\|} \right)^{a_k}, \\ &\quad E \text{ is } (I, \nu_{k+1,p})\text{-thin}\}, \\ \mathcal{C}_{k,p} &= \mathcal{C}_{k+1,p} \cup \mathcal{E}_{k,p}. \end{aligned}$$

We also define a sequence $(c_k)_k$ by setting $c_0 = 8$ and $c_{k+1} = 4^{a_k} c_k$, for all k .

Let us compare our construction with the one from [17]. First, Talagrand starts by setting $\mathcal{C}_{p,p} = \mathcal{D}$, for some suitable family \mathcal{D} . This was done in order to ensure that all the submeasure are pathological, but it is not really necessary since we will have an explicit reason why our submeasures are not uniformly exhaustive. The main difference is that in the definition of the $\mathcal{E}_{k,p}$, instead of the cardinality of I we use $\|I\|$, where $\|\cdot\|$ is our given admissible norm. Of course, the notion of an admissible norm was tailor-made so that analogs of the key arguments from [17] would go through. The upshot is that by varying our norm $\|\cdot\|$ we obtain uncountably many essentially different examples of exhaustive non uniformly exhaustive submeasures. For the remainder of this section we prove some technical lemmas and show that $\nu_{k,p}(N_u) \geq 8$, for all $k \leq p$ and all $u \in \mathcal{P}$ such that $\|\text{dom}(u)\| \leq 1$.

Lemma 3.8. Let k, p and m be integers with $k \leq p$. Suppose $(X, I, w) \in \mathcal{C}_{k,p}$ and $A \in \mathcal{A}_m$. Suppose that $n > m$ and $I' = I \cap [m, n)$ is non-empty. Set $X' = [\pi_A^{-1}(X \cap A)]_n$. Then $(X', I', w') \in \mathcal{C}_{k,p}$, where $w' = w \cdot \left(\frac{\|I\|}{\|I'\|} \right)^{a_k}$.

Proof. By the definition of $\mathcal{C}_{k,p}$, there is some r with $k \leq r < p$ such that $(X, I, w) \in \mathcal{E}_{r,p}$. Let us fix such r and let us show that X' is $(I', \nu_{r+1,p})$ -thin. Since we only used $X \cap A$ in the definition of X' , we may assume that $X \subseteq A$. Let $i, j \in I'$ be such that $i < j$. We need to show that for every $A_1 \in \mathcal{A}_i$ there is $H \in \mathcal{B}_j$ such that $H \subseteq A_1 \setminus X'$ and $\nu_{r+1,p}(\pi_{A_1}^{-1}(H)) > 1$. Now, if $A_1 \subseteq A$ this follows from the fact that X is $(i, j, \nu_{r+1,p})$ -thin. If $A_1 \cap A = \emptyset$ let $A_2 = \pi_A(A_1)$. Then, as before, we can find $H \subseteq A_2 \setminus X$ such that $H \in \mathcal{B}_j$ and $\nu_{r+1,p}(\pi_{A_2}^{-1}(H)) > 1$. Let $H' = \pi_{A_2}^{-1}(H) \cap A_1 \in \mathcal{B}_j$. Since $\pi_{A_1}^{-1}(H') = \pi_{A_2}^{-1}(H)$ we also have $\nu_{r+1,p}(\pi_{A_1}^{-1}(H')) > 1$. Moreover, we have $H' \cap \pi_A^{-1}(X) = \emptyset$. Indeed, if $x \in H' \cap \pi_A^{-1}(X)$ we would have that $\pi_{A_2}(x) \in X \cap H = \emptyset$. Since $H' \in \mathcal{B}_j$ we also have $H' \in \mathcal{B}_n$. Therefore, $H' \cap [\pi_A^{-1}(X)]_n = \emptyset$ i.e. $H' \cap X' = \emptyset$. Finally, let us check the weight

condition. Since, by definition, $w \geq 2^{-r} \left(\frac{M_r}{\|I'\|} \right)^{a_r}$ and $k \leq r$, we get

$$\begin{aligned} w' &= w \cdot \left(\frac{\|I\|}{\|I'\|} \right)^{a_k} \\ &\geq w \cdot \left(\frac{\|I\|}{\|I'\|} \right)^{a_r} \\ &\geq 2^{-r} \left(\frac{M_r}{\|I'\|} \right)^{a_r}. \end{aligned}$$

This proves that $(X', I', w') \in \mathcal{E}_{rp} \subseteq \mathcal{C}_{rp} \subseteq \mathcal{C}_{kp}$, as desired. \square

For the purpose of the following lemma we shall extend our previous notation and if \mathcal{D} is a subset of $[\mathbb{N}]^{<\omega} \times \mathbb{R}^+$ we shall write

$$w(\mathcal{D}) = \sum_{(I,w) \in \mathcal{D}} w.$$

Lemma 3.9. *Suppose $t \geq 5$ and $J_0 < \dots < J_{s-1}$ are finite sets with $\|J_i\| \geq t$, for all $i < s$. Suppose \mathcal{F} is a finite subset of $[\mathbb{N}]^{<\omega} \times \mathbb{R}^+$ and let $a = w(\mathcal{F})$. Then we can find integers $m_i, n_i \in J_i$, with $m_i < n_i$ for all $i < s$, such that:*

- (1) $\|J_i \cap [m_i, n_i]\| \geq 3$, for all $i < s$,
- (2) if we let $W = \bigcup_{i < s} [m_i, n_i]$ and $\mathcal{D} = \{(I, w) \in \mathcal{F} : \|I \setminus W\| < \frac{1}{2}\|I\|\}$ then

$$w(\mathcal{D}) \leq \frac{3a}{t-4}.$$

Proof. Let $c = \lfloor \frac{t-1}{3} \rfloor$. For each $i < s$, we pick an increasing sequence $\{m_{i,l} : l \leq c\}$ of elements of J_i such that $\|J_i \cap [m_{i,l}, m_{i,l+1}]\| = 3$, for all $l < c$. Given $l < c$, let $W_l = \bigcup_{i < s} [m_{i,l}, m_{i,l+1})$ and let

$$\mathcal{D}_l = \{(I, w) \in \mathcal{F} : \|I \setminus W_l\| < \frac{\|I\|}{2}\}.$$

Since the W_l are pairwise disjoint and $\|\cdot\|$ is subadditive, it follows that the \mathcal{D}_l are pairwise disjoint. Therefore, we get that

$$w(\mathcal{D}_0) + \dots + w(\mathcal{D}_{c-1}) \leq w(\mathcal{F}) = a.$$

By using the fact that the minimum of a finite sequence is less than or equal to its average, we conclude that there exists $l < c$ such that:

$$w(\mathcal{D}_l) \leq \frac{a}{c} \leq \frac{a}{(t-1)/3-1} = \frac{3a}{t-4}.$$

Therefore, we can let $m_i = m_{i,l}$ and $n_i = m_{i,l+1}$, for all $i < s$, and this satisfies the conclusion of the lemma. \square

In the next proposition we adapt the argument of Theorem 5.1 from [17].

Proposition 3.10. *Suppose k and p are integers with $k \leq p$. Then $\nu_{k,p}(N_u) \geq c_k$, for every $u \in \mathcal{P}$ with $\|\text{dom}(u)\| \leq 1$. In particular, $\nu_{k,p}(T) \geq c_k$.*

Proof. Let us fix p and prove the statement by backwards induction on k . If $k = p$ the statement is obvious since $\nu_{p,p}(X) = +\infty$, for every non empty $X \in \mathcal{B}$. Thus, let us assume $k < p$, the inequality holds for $k + 1$, and let us check that it holds for k . Fix $u \in \mathcal{P}$ with $\|\text{dom}(u)\| = 1$ and a finite $\mathcal{F} \subseteq \mathcal{C}_{k,p}$ with $w(\mathcal{F}) < c_k$. We have to show that $N_u \not\subseteq X(\mathcal{F})$.

To begin let us fix $\mathcal{F}_1 \subseteq \mathcal{E}_{k,p}$, $\mathcal{F}_2 \subseteq \mathcal{C}_{k+1,p}$ such that $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$. Let $s = |\mathcal{F}_1|$. Since for $(X, I, w) \in \mathcal{F}_1$ we have $w \geq 2^{-k}$, we get $s \leq 2^k c_k \leq 2^{k+4}$. On the other hand, for every such (X, I, w) we have

$$2^{-k} \left(\frac{M_k}{\|I\|} \right)^{a_k} \leq w \leq c_k \leq 2^4,$$

so an easy calculation gives us $\|I\| \geq 2^{k+8}s$. Using Lemma 2.15, we can enumerate \mathcal{F}_1 as $\{(X_l, I_l, w_l) : l < s\}$, and find sets J_0, \dots, J_{s-1} , such that $J_l \subseteq I_l$ and $\|J_l\| \geq 2^{k+8}$, for every $l < s$, and moreover $J_0 < \dots < J_{s-1}$. Applying Lemma 3.9, where $a = w(\mathcal{F}_2) \leq 2^4$, we can find $m'_l, n'_l \in J_l$ with $m'_l < n'_l$, for $l < s$, so that $\|J_l \cap [m'_l, n'_l]\| = 3$ and, if we let,

$$W' = \bigcup_{l < s} [m'_l, n'_l] \text{ and } \mathcal{F}'_3 = \{(X, I, w) \in \mathcal{F}_2 : \|I \setminus W'\| < \frac{1}{2}\|I\|\},$$

then

$$w(\mathcal{F}'_3) \leq \frac{3 \cdot 2^4}{2^{k+8} - 4} \leq \frac{12}{2^{k+6} - 1} \leq \frac{1}{4}.$$

Since $\|\text{dom}(u)\| = 1$, by Lemma 2.16 we can find consecutive elements $m_l, n_l \in J_l \cap [m'_l, n'_l]$ such that $\text{dom}(u) \cap [m_l, n_l] = \emptyset$, for every $l < s$. Now, let

$$W = \bigcup_{i < s} [m_i, n_i], \quad \mathcal{F}_3 = \{(X, I, w) \in \mathcal{F}_2 : \|I \setminus W\| < \frac{1}{2}\|I\|\}, \quad \mathcal{F}_4 = \mathcal{F}_2 \setminus \mathcal{F}_3.$$

Since $\mathcal{F}_3 \subseteq \mathcal{F}'_3$ we have that $w(\mathcal{F}_3) \leq w(\mathcal{F}'_3) \leq 1/4$. Since $I = (I \cap W) \cup (I \setminus W)$, for $(X, I, w) \in \mathcal{F}_3$, we also get $\|I \cap W\| \geq \frac{1}{2}\|I\|$. For $l < s$ let

$$\mathcal{F}_{3,l} = \{(X, I, w) \in \mathcal{F}_3 : \|I \cap [m_l, n_l]\| \geq 2^{-k-5}\|I\|\}.$$

Since $s \leq 2^{k+4}$ and $\|I \cap W\| \geq \frac{1}{2}\|I\|$, we get $\mathcal{F}_3 = \bigcup_{l < s} \mathcal{F}_{3,l}$.

Claim 3.11. *For every $l < s$ and $A \in \mathcal{A}_{m_l}$, there is $A' \in \mathcal{A}_{n_l}$ with $A' \subseteq A \setminus (X_l \cup X(\mathcal{F}_{3,l}))$.*

Proof. Let $A \in \mathcal{A}_{m_l}$. Using Lemma 3.8 for $C = [\pi_A^{-1}(A \cap X(\mathcal{F}_{3,l}))]_{n_l} \in \mathcal{B}_{n_l}$ we get:

$$\nu_{k+1,p}(C) \leq 2w(\mathcal{F}_{3,l}) \leq \frac{1}{2}.$$

Since $(X_l, I_l, w_l) \in \mathcal{E}_{k,p}$, X_l is $(m_l, n_l, \nu_{k+1,p})$ -thin. Therefore, for $K = T \setminus [\pi_A^{-1}(A \cap X_l)]_{n_l}$, we have $\nu_{k+1,p}(K) > 1$. Since both C and K belong to \mathcal{B}_{n_l} , there is an atom $A' \in \mathcal{A}_{n_l}$, such that $A' \subseteq K \setminus C$. Since $\pi_A^{-1}(K) = K$ and $\pi_A^{-1}(C) = C$, we may find such A' which is contained in A . \square

Now define a function $\Gamma : T \rightarrow T$ as follows. For $x \in T$ and $j \in \mathbb{N} \setminus W$, we let $\Gamma(x)(j) = x(j)$. Let us consider some $x \in T$ and an interval $[m_l, n_l)$ and suppose $\Gamma(x) \upharpoonright m_l$ has been defined. Let $v = \Gamma(x) \upharpoonright m_l$. Applying Claim 3.11 to $A = N_v$, we find

some $A' \in \mathcal{A}_{n_l}$ with $A' \subseteq A \setminus (X_l \cup X(\mathcal{F}_{3,l}))$. Let v' be such that $A' = N_{v'}$. We then let $\Gamma(x) \upharpoonright n_l = v'$. Notice that we have assured that $\Gamma(T)$ is disjoint from $X(\mathcal{F}_1) \cup X(\mathcal{F}_3)$.

Claim 3.12. $\nu_{k+1,p}(\Gamma^{-1}[X(\mathcal{F}_4)]) < c_{k+1}$.

Proof. Let $(X, I, w) \in \mathcal{F}_4$. We want to estimate $\nu_{k+1,p}(\Gamma^{-1}[X])$. There is an r such that $k+1 \leq r < p$ and $(X, I, w) \in \mathcal{E}_{r,p}$. We will show first that for $m, n \in I$, $m < n$, if $[m, n)$ is disjoint from W , then $\Gamma^{-1}[X]$ is $(m, n, \nu_{r+1,p})$ -thin. In order to see this, let $A \in \mathcal{A}_m$, and let A' be an atom in \mathcal{A}_m such that $\Gamma(A) \subseteq A'$. Since X is $(m, n, \nu_{r+1,p})$ -thin, within A' we replicate the thinness of X in A' inside A to establish the thinness of $\Gamma^{-1}[X]$. More precisely, since X is $(m, n, \nu_{r+1,p})$ -thin, within A' there exists $H \in \mathcal{B}_n$ such that $H \subseteq A'$, $H \cap X = \emptyset$, and $\nu_{r+1,p}(\pi_{A'}^{-1}(H)) > 1$. Since H is disjoint from X , we also have that $\Gamma^{-1}[H]$ is disjoint from $\Gamma^{-1}[X]$. From $\Gamma(\pi_A(\pi_{A'}^{-1}(H))) \subseteq H$, we get

$$\pi_{A'}^{-1}(H) \subseteq \pi_A^{-1}(\Gamma^{-1}[H]) \subseteq \pi_A^{-1}(H).$$

We assumed that $\nu_{r+1,p}(\pi_{A'}^{-1}(H)) > 1$, so $\nu_{r+1,p}(\pi_A^{-1}(H)) > 1$, which proves that $\Gamma^{-1}[X]$ is $(m, n, \nu_{r+1,p})$ -thin.

We now have to deal with pairs of elements of I that are separated by W . Since thinness is monotone in the second coordinate, this is a problem only for $m, n \in I$ with $m < n$ such that m is the last element in I preceding some interval $[m_l, n_l)$. We saw in the beginning of the proof that $\|I\| \geq 2^{k+8}s \geq 4s$, so $s \leq \|I\|/4$. From the definition of \mathcal{F}_4 we have $\|I \setminus W\| \geq \|I\|/2$. For every $l < s$, let i_l be the largest element of I below m_l . Then, for:

$$I' = I \setminus (W \cup \{i_l : l < s\}),$$

we have that $\|I'\| \geq \|I\|/4$. We now have that $\Gamma^{-1}[X]$ is $(I', \nu_{r+1,p})$ -thin. We also need a bound for the norm. Let

$$w' = w \left(\frac{\|I\|}{\|I'\|} \right)^{a_k} \leq 4^{a_k} w \leq 4^{a_k} w.$$

We have that $(\Gamma^{-1}[X], I', w') \in \mathcal{E}_{r,p} \subseteq \mathcal{C}_{k+1,p}$. This establishes $\nu_{k+1,p}(\Gamma^{-1}[X]) \leq 4^{a_k} w$. Now we have,

$$\nu_{k+1,p}(\Gamma^{-1}[X(\mathcal{F}_4)]) \leq 4^{a_k} w(\mathcal{F}_4) < 4^{a_k} c_k = c_{k+1},$$

as needed. \square

Now, by Claim 3.12 and the inductive assumption, pick some $z \in N_u \setminus \Gamma^{-1}[X(\mathcal{F}_4)]$. Since $W \cap \text{dom}(u) = \emptyset$ and $\Gamma(x)(j) = x(j)$, for every $x \in T$ and $j \notin W$, it follows that $\Gamma(z) \in N_u$. On the other hand, we have already shown that $\Gamma(T)$ is disjoint from $X(\mathcal{F}_1)$ and $X(\mathcal{F}_3)$. Since $z \in N_u \setminus \Gamma^{-1}[X(\mathcal{F}_4)]$ it follows that

$$\Gamma(z) \notin X(\mathcal{F}_1) \cup X(\mathcal{F}_3) \cup X(\mathcal{F}_4) = X(\mathcal{F}).$$

This completes the proof of Proposition 3.10. \square

Definition 3.13. Let \mathcal{U} be a non principal ultrafilter on \mathbb{N} . For $k \in \mathbb{N}$ and $E \in \mathcal{B}$, we define $\nu_k(E) = \lim_{p \rightarrow \mathcal{U}} \nu_{k,p}(E)$. We write ν for ν_0 .

Proposition 3.14. For every integer k , ν_k is a submeasure and $\nu_k(N_u) \geq 8$, for every $u \in \mathcal{P}$ with $\|\text{dom}(u)\| = 1$. In particular, ν_k is not uniformly exhaustive.

Proof. First note that ν_k is a submeasure as an ultrafilter limit of submeasures. Let $u \in \mathcal{P}$ be such that $\|\text{dom}(u)\| = 1$. By Proposition 3.10, $\nu_{k,p}(N_u) \geq c_k \geq 8$, for every $p \geq k$. Therefore $\nu_k(N_u) \geq 8$, as well. Given an integer n and $i < 2^n$, let $u_i = \{(n, i)\}$. Since $\|\{n\}\| = 1$, we have that $\nu_k(N_{u_i}) \geq 8$, for all $i < 2^n$. Since the family $\{N_{u_i} : i < 2^n\}$ is pairwise disjoint, for every n , it follows that ν_k is not 8-uniformly exhaustive. \square

For a countable ordinal $\alpha > 0$, let us write ν^α for the submeasure ν constructed from the admissible norm $\|\cdot\|_\alpha$ from Definition 2.11. By Lemma 2.14 and Proposition 3.10, we have the following immediate corollary.

Corollary 3.15. *The exhaustivity rank of ν^α is at least ω^α , for $0 < \alpha < \omega_1$.* \square

4. EXHAUSTIVITY

In this section we still work with a given admissible norm $\|\cdot\|$ and the submeasures $\nu_{k,p}$ given by Definition 3.7. We now turn to the proof that the limit submeasures ν_k are exhaustive. We organize our argument in a way to also be able to provide upper bounds on their exhaustivity ranks.

Lemma 4.1. *For every k and p with $k \leq p$, the submeasure $\nu_{k,p}$ has the covering property.*

Proof. Suppose m is an integer, $E \in \mathcal{B}$ does not depend on coordinates $< m$ and $\nu_{k,p}(E) < 2$. Let $n = n(E)$. We need to construct an m -covering sequence for E . Fix some $\mathcal{F} \subseteq \mathcal{C}_{k,p}$ with $E \subseteq X(\mathcal{F})$ and $w(\mathcal{F}) < 2$. For $r > m$ we let:

$$\mathcal{F}_r = \{(X, I, w) \in \mathcal{F} : \|I \cap [m, r]\| < \frac{1}{2}\|I\| \text{ \& } \|I \cap [m, r]\| \geq \frac{1}{2}\|I\|\}.$$

We also let

$$\mathcal{F}' = \{(X, I, w) \in \mathcal{F} : \|I \cap m\| \geq \frac{1}{4}\|I\|\}.$$

We use Lemma 3.8 to get a set $B \in \mathcal{B}_m$ such that $X(\mathcal{F}') \subseteq B$ and

$$\nu_{k,p}(B) \leq 4^{a_k} w(\mathcal{F}') \leq 4.$$

Since $\nu_{k,p}(T) \geq 8$, $B \neq T$, so there exists $A_m \in \mathcal{A}_m$ such that $A_m \cap X(\mathcal{F}') = \emptyset$. For $(X, I, w) \in \mathcal{F}_r$, let $X' = [\pi_{A_p}^{-1}(X \cap A_m)]_r$ and $I' = I \cap [m, r]$. Note that $X \cap A_m \subseteq X'$. By Lemma 3.8 again, we can find some $w' \leq 2w$ such that $(X', I', w') \in \mathcal{C}_{k,p}$. Let \mathcal{F}'_r be the collection of triples (X', I', w') obtained in this way.

Claim 4.2. *For every j such that $m < j \leq n$, we have*

$$\text{int}_j(E) \subseteq \bigcup_{m < r \leq j} X(\mathcal{F}'_r).$$

Proof. Note that $\text{int}_j(E)$ and the sets $X(\mathcal{F}'_r)$, for $m < r \leq j$, depend only on the coordinates in the interval $[m, j]$. Therefore, if the inclusion does not hold we can find $A \in \mathcal{A}_j$ such that $A \subseteq A_m \cap E$, yet $A \cap X(\mathcal{F}'_r) = \emptyset$, for all r with $m < r \leq j$. Since $A_p \cap X(\mathcal{F}_r) \subseteq A_p \cap X(\mathcal{F}'_r)$, it follows that $A \cap X(\mathcal{F}_r) = \emptyset$, for every r with $m < r \leq j$. Finally, we have that $A_m \cap X(\mathcal{F}') = \emptyset$. All this means that $A \subseteq X(\mathcal{F}'')$, where

$$\mathcal{F}'' = \mathcal{F} \setminus (\mathcal{F}' \cup \bigcup_{r \leq j} \mathcal{F}_r).$$

Note that if $(X, I, w) \in \mathcal{F}''$ then $\|I \setminus j\| \geq \frac{1}{4}\|I\|$. By applying Lemma 3.8 one more time, we can find a set X^* , covering X and depending only on coordinates $\geq j$, and some $w^* \leq 2w$ such that, letting $I^* = I \setminus j$, we have $(X^*, I^*, w^*) \in \mathcal{C}_{k,p}$. Let \mathcal{F}^* be the set of triples (X^*, I^*, w^*) obtained in this way. Then $X(\mathcal{F}^*)$ depends only on coordinates $\geq j$ and contains $X(\mathcal{F}'')$, and

$$w(\mathcal{F}^*) \leq 2w(\mathcal{F}) \leq 4.$$

Since $A \subseteq X(\mathcal{F}^*)$ and $A \in \mathcal{B}_j$, it follows that $X(\mathcal{F}^*)$ covers all of T . This implies that $\nu_{k,p}(T) \leq 4$, a contradiction. \square

For $m < r \leq n$, the set $X(\mathcal{F}'_r)$ depends only on coordinates in the interval $[m, r)$ and

$$\nu_{k,p}(X(\mathcal{F}'_r)) \leq w(\mathcal{F}'_r) \leq 2w(\mathcal{F}_r).$$

Therefore, we get that:

$$\sum_{m < r \leq n} \nu_{k,p}(X(\mathcal{F}'_r)) \leq \sum_{m < r \leq n} 2w(\mathcal{F}_r) \leq 2w(\mathcal{F}) \leq 4.$$

It follows that if we let $C_r^m(E) = X(\mathcal{F}'_r)$, for $m < r \leq n$, the resulting sequence is an m -covering sequence for E . \square

Lemma 4.3. *The submeasure ν_k has the covering property, for every k .*

Proof. Suppose m is an integer, E is a set in \mathcal{B} that does not depend on coordinates $< m$, and $\nu_k(E) < 2$. Let $n = n(E)$. By the definition of ν_k , the set $U = \{p : \nu_{k,p}(E) < 2\}$ belongs to \mathcal{U} . For each $p \in U$, fix an m -covering sequence $\{C_{r,p}^m(E) : m < r \leq n\}$ of E with respect to $\nu_{k,p}$. Since \mathcal{U} is a ultrafilters and the set of all possible such sequences is finite, there is a fixed sequence $\{C_r^m(E) : m < r \leq n\}$ such that

$$V = \{p \in U : C_{r,p}^m(E) = C_r^m(E), \text{ for all } m < r \leq n\} \in \mathcal{U}.$$

It is clear now that $\{C_r^m(E) : m < r \leq n\}$ is an m -covering sequence for E with respect to ν_k . \square

Lemma 4.4. *Let k be an integer and $(E_i)_i$ a sequence of sets in \mathcal{B} not depending on coordinates $< m$ such that $\nu_k(\bigcup_{i < n} E_i) < 2$, for every n . Then, for every $\eta > 0$, there is $C \in \mathcal{B}$ that does not depend on coordinates $< m$ such that $\nu_k(C) \leq 4$ and $\nu_k(E_i \setminus C) \leq \eta$, for all i .*

Proof. For each n , let $G_n = \bigcup_{i < n} E_i$ and let $\vec{C}(G_n)$ be an m -covering sequence for G_n . Since, for each $l > m$, there are only finitely many possibilities for $\vec{C}(G_n) \upharpoonright [m, l)$, by König's Lemma there is an infinite sequence $\vec{C} = \{C_j : j > m\}$ such that, for every $l > m$, there are arbitrary large n such that $\vec{C} \upharpoonright l = \vec{C}(G_n) \upharpoonright l$. It follows that

$$\bigcup_i E_i \subseteq \bigcup_{m < j} C_j \text{ and } \sum_{m < j} \nu_k(C_j) \leq 4.$$

Let l be such that $\sum_{l \leq j} \nu_k(C_j) \leq \eta$. Then the set $C = \bigcup \{C_j : m < j < l\}$ satisfies the conclusion of the lemma. \square

Lemma 4.5. *Let k and m be integers and $\eta > 0$. Suppose $(E_i)_i$ is a pairwise disjoint sequence of sets in \mathcal{B} . Then there is $n > m$ and $B \in \mathcal{B}_n$, B is (m, n, ν_k) -thin, and $\limsup_{i \rightarrow \infty} \nu_k(E_i \setminus B) \leq \eta$.*

Proof. Let $\eta' = \eta/|\mathcal{A}_m|$. For all $A \in \mathcal{A}_m$ we define $H(A) \subseteq A$ such that

$$\nu_k(\pi_A^{-1}(H(A))) > 1 \text{ and } \limsup_{i \rightarrow \infty} \nu_k(E_i \cap H(A)) \leq \eta'.$$

Case 1. There exists an integer r such that $\nu_k(\pi_A^{-1}(A \cap \bigcup_{i < r} E_i)) > 1$.

We let $H(A) = A \cap \bigcup_{i < r} E_i$. Note that $E_i \cap H(A) = \emptyset$, for all $i \geq r$.

Case 2. $\nu_k(\pi_A^{-1}(A \cap \bigcup_{i < r} E_i)) \leq 1$, for every integer r .

Since the sets $\pi_A^{-1}(E_i)$ do not depend on coordinates $< m$, by Lemma 4.4, we can find $C \in \mathcal{B}$ that does not depend on coordinates $< m$ such that $\nu_k(C) \leq 4$ and

$$\limsup_{i \rightarrow \infty} \nu_k(\pi_A^{-1}(E_i) \setminus C) \leq \eta'.$$

If we take $H(A) = A \setminus C$, then $C = T \setminus \pi_A^{-1}(H(A))$. Since $\nu_k(T) \geq 8$, we have $\nu_k(\pi_A^{-1}(H(A))) > 1$. Since π_A is the identity on A , we have that, for all i ,

$$E_i \cap H(A) \subseteq \pi_A^{-1}(E_i) \setminus C,$$

and so

$$\limsup_{i \rightarrow \infty} \nu_k(E_i \cap H(A)) \leq \limsup_{i \rightarrow \infty} \nu_k(\pi_A^{-1}(E_i) \setminus C) \leq \eta'.$$

Now, let $D(A) = A \setminus H(A)$, let $B = \bigcup \{D(A) : A \in \mathcal{A}_m\}$, and let n be the least such that $B \in \mathcal{B}_n$. Then B and n are as required. \square

Lemma 4.6. *The submeasure ν_k has the s -thinness property, for all $s \geq k$.*

Proof. Fix some $s \geq k$, and suppose $\|I\| = M_s$ and X is (I, ν_k) -thin. Since this property of X depends on the fact that the submeasure of finitely many sets is > 1 and \mathcal{U} is non principal, it follows that:

$$U = \{p \geq s+1 : X \text{ is } (I, \nu_{k,p})\text{-thin}\} \in \mathcal{U}.$$

Fix some $p \in U$. Since $\nu_{k,p} \leq \nu_{s+1,p}$, we have that X is also $(I, \nu_{s+1,p})$ -thin, and hence $(X, I, 2^{-s}) \in \mathcal{E}_{s,p}$. It follows that $\nu_{k,p}(X) \leq 2^{-s}$. Since this holds for all $p \in U$ and $\nu_k = \lim_{p \rightarrow \mathcal{U}} \nu_{k,p}$, we conclude that $\nu_k(X) \leq 2^{-s}$, as desired. \square

Proposition 4.7. *For every integer k , the submeasure ν_k is exhaustive.*

Proof. Fix k and suppose $(E_i)_i$ is a pairwise disjoint sequence of sets in \mathcal{B} . Fix some $s \geq k$ and $\epsilon > 0$. Starting with $n_0 = 0$, we use Lemma 4.5 repeatedly to construct an increasing sequence of integers $(n_l)_l$ and sets $B_l \in \mathcal{B}_{n_{l+1}}$ such that, for all l , B_l is (n_l, n_{l+1}, ν_k) -thin, and

$$\limsup_{i \rightarrow \infty} \nu_k(E_i \setminus B_l) \leq \frac{\epsilon}{2^{l+1}}.$$

Let $I_l = \{n_0, n_1, \dots, n_l\}$. Since our norm $\|\cdot\|$ is unbounded, there is l such that $\|I_l\| = M_s$. Let $B = \bigcap_{i < l} B_i$. Then the set B is (I_l, ν_k) -thin. By the s -thinness property of ν_k we have that $\nu_k(B) \leq 2^{-s}$. Now, by the subadditivity of ν_k we have:

$$\limsup_{i \rightarrow \infty} \nu_k(E_i \setminus B) \leq \sum_{j < l} \limsup_{i \rightarrow \infty} \nu_k(E_i \setminus B_j) \leq (1 - \frac{1}{2^{l+1}})\epsilon.$$

Since $s \geq k$ and $\epsilon > 0$ were arbitrary, it follows that $\limsup_{i \rightarrow \infty} \nu_k(E_i) = 0$. This completes the proof that ν_k is exhaustive. \square

Now, by combining Proposition 4.7 and Corollary 3.15 we obtain our main result.

Theorem 4.8. *There are exhaustive submeasures on \mathcal{B} of arbitrary high countable exhaustivity rank.* \square

Let \mathcal{E} denote the set of all exhaustive submeasure on \mathcal{B} . It is easy to see that \mathcal{E} is a co-analytic subset of $[0, +\infty]^\mathcal{B}$ with the product topology. We now have the following corollary.

Corollary 4.9. *The set \mathcal{E} of exhaustive submeasures on \mathcal{B} is not Borel.*

Proof. Indeed, the function $\nu \mapsto \text{rank}(\nu)$ is clearly a Π_1^1 -rank. Since, by Theorem 4.8, this function is unbounded below ω_1 , by the rank method (see [10], page 288) the set \mathcal{E} is not Borel. \square

Suppose ν is strictly positive exhaustive submeasure on \mathcal{B} . In the standard way we define a metric ρ on \mathcal{B} : $\rho(E, F) = \nu(E \triangle F)$, for $E, F \in \mathcal{B}$. We use it to obtain a metric completion $\bar{\mathcal{B}}$ of \mathcal{B} . The continuous extension $\bar{\nu}$ of ν to $\bar{\mathcal{B}}$ is a strictly positive continuous submeasure of exhaustivity rank the same as ν . Since, by [11] any two continuous submeasures on a Maharam algebra \mathcal{M} are absolutely continuous with respect to each other, the exhaustivity rank is an algebraic invariant of \mathcal{M} . Therefore, from Theorem 4.8 we have the following corollary.

Corollary 4.10. *There are uncountably many pairwise non isomorphic separable atomless Maharam algebras.* \square

5. BOUNDING THE EXHAUSTIVITY RANKS

As mentioned in the introduction Fremlin [4] showed that the exhaustivity rank of Talagrand's submeasure from [17] is at most ω^{ω^2} . In this section we give bounds on the exhaustivity rank of our submeasures. If one wishes, one can then produce an explicit ω_1 -sequence of pairwise non isomorphic Maharam algebras. Thus, suppose ν is a submeasure on \mathcal{B} satisfying the covering property and such that $\nu(T) \geq 8$. Suppose that $\|\cdot\|$ is an admissible norm, N is an integer and ν satisfies the N -thinness property relative to $\|\cdot\|$. Recall that this means that there is an integer M_N such that that $\nu(X) \leq 2^{-N}$, for every set X which is (I, ν) -thin, for some I with $\|I\| = M_N$. Let $\mathcal{S} = \{F \in [\mathbb{N}]^{<\omega} : \|F\| < M_N\}$ and let $\beta = \rho(\mathcal{S})$. Recall that this means that β is the least ordinal for which Player I has a winning strategy in the game $\mathcal{G}_\beta(\mathcal{S})$ from Definition 2.4. For any $\epsilon > 0$, we give an explicit bound on the $(2^{-N} + \epsilon)$ -exhaustivity rank of ν .

We start by making some definitions. Suppose m is an integer and $A \in \mathcal{A}_m$. If $X \in \mathcal{B}$ we let $\nu(X|A)$ denote the relative submeasure of X with respect to A , i.e. $\nu(\pi_A^{-1}(X))$. Note that $\nu(X|A) = \nu(X \cap A|A)$. Suppose now $n > m$ and $\vec{C} = \{C_r : m < r \leq n\}$ is a sequence such that $C_r \subseteq A$ and $C_r \in \mathcal{B}_r$, for all $m < r \leq n$. We let

$$w(\vec{C}|A) = \sum_{m < r \leq n} \nu(C_r|A).$$

Definition 5.1. Suppose $m < n$ and $A \in \mathcal{A}_m$. We let $\mathcal{C}_{m,n}(A)$ denote the collection of all sequences $\vec{C} = \{C_r : m < r \leq n\}$ such that $C_r \subseteq A$, $C_r \in \mathcal{B}_r$, for all $m < r \leq n$, and $w(\vec{C}|A) \leq 4$. We let $\mathcal{C}_m(A) = \bigcup \{\mathcal{C}_{m,n}(A) : m < n\}$.

Suppose now $m < n \leq p$, $A \in \mathcal{A}_m$, $\vec{C} \in \mathcal{C}_{m,n}(A)$ and $\vec{D} \in \mathcal{C}_{m,p}(A)$. We say that \vec{D} is an *extension* of \vec{C} if $\vec{D} \upharpoonright (m, n] = \vec{C}$. If $\delta > 0$ we say that \vec{D} is a δ -*proper extension* of \vec{C} if \vec{D} is an extension of \vec{C} and $w(\vec{D}|A) \geq w(\vec{C}|A) + \delta$.

Lemma 5.2. Let m be an integer and $A \in \mathcal{A}_m$. Suppose $E \subseteq A$ and $\nu(E|A) < 2$. Let $n > m$ be such that $E \in \mathcal{B}_n$. Then there is $\vec{C} \in \mathcal{C}_{m,n}(A)$ such that $E \subseteq \bigcup \vec{C}$.

Proof. Let $E' = \pi_A^{-1}(E)$. Then E' does not depend on coordinates $< m$ and $\nu(E') < 2$. By the m -covering property, we can fix an m -covering sequence $\{C'_r : m < r \leq n\}$ of E' . Let $C_r = C'_r \cap A$, for all $m < r \leq n$. Then $\vec{C} = \{C_r : m < r \leq n\}$ is as required. \square

Definition 5.3. Let α be an ordinal, m an integer, $A \in \mathcal{A}_m$, and $\delta > 0$. The game $\mathcal{G}(\alpha, A, \delta)$ is played between two players I and II as follows.

$$\begin{array}{ccccccc} \text{I :} & \alpha_0, C_0 & \alpha_1, C_1 & \cdots & \alpha_n, C_n & \cdots \\ \hline \text{II :} & E_0 & E_1 & \cdots & E_n & \cdots \end{array}$$

Player I plays ordinals $\leq \alpha$ such that $\alpha_n \leq \alpha_{n-1}$, and clopen sets $C_n \subseteq A$ such that $\nu(C_n|A) \leq 4$. Player II plays clopen sets $E_n \subseteq A$. Player I is required to play $\alpha_{n+1} < \alpha_n$ if $\nu(\bigcup_{i < n} E_i|A) < 2$ and, either $\nu(\bigcup_{i \leq n} E_i|A) \geq 2$ or $\nu(E_n \setminus C_n|A) \geq \delta$. In other case Player I is allowed to play $\alpha_{n+1} = \alpha_n$. Player I wins if he can keep playing indefinitely by following these rules.

Lemma 5.4. Let m be an integer, $A \in \mathcal{A}_m$ and $\delta > 0$. Let $k = \lceil 4/\delta \rceil$. Then Player I has a winning strategy in $\mathcal{G}(\omega^{k+1}, A, \delta)$.

Proof. For $\vec{C} \in \mathcal{C}_m(A)$, let $k(\vec{C})$ be the least integer l such that $(l+1) \cdot \delta > 4 - w(\vec{C}|A)$. To begin, Player I plays $(\omega^{k+1}, \emptyset)$. As long as $\nu(\bigcup_{i < n} E_i|A) < 2$, Player I plays ordinals $\alpha_n > 0$. At stage $n > 0$, if $\nu(\bigcup_{i < n} E_i|A) < 2$, by Lemma 5.2 there is $\vec{C} \in \mathcal{C}_m(A)$ such that $\bigcup_{i < n} E_i \subseteq \bigcup \vec{C}$. On the side Player I keeps an integer $p(n)$ such that $E_i \in \mathcal{B}_{p(n)}$, for all $i < n$, and a finite family $\mathcal{D}_n \subseteq \mathcal{C}_m(A)$ such that every $\vec{C} \in \mathcal{C}_{m,p(n)}(A)$ such that $\bigcup_{i < n} E_i \subseteq \bigcup \vec{C}$ extends a member of \mathcal{D}_n . Given these objects, let us define α'_n to be the natural sum of the ordinals $\omega^{k(\vec{C})}$, for $\vec{C} \in \mathcal{D}_n$, and let $\alpha_n = \alpha'_n + 1$. Player I picks some $\vec{C}_n \in \mathcal{D}_n$, sets $C_n = \bigcup \vec{C}_n$ and plays the pair (α_n, C_n) . Suppose Player II responds by playing some E_n . If $\nu(\bigcup_{i \leq n} E_i|A) < 2$ and $\nu(E_n \setminus C_n|A) < \delta$, Player I simply repeats his previous move, i.e. he sets $(\alpha_{n+1}, C_{n+1}) = (\alpha_n, C_n)$. He also sets $\mathcal{D}_{n+1} = \mathcal{D}_n$. If $\nu(\bigcup_{i \leq n} E_i|A) \geq 2$, Player I sets $\alpha_{n+1} = 0$ and $C_{n+1} = \emptyset$. After that there are no requirements for him, so he keeps repeating this move indefinitely. Suppose now that $\nu(E_n \setminus C_n|A) \geq \delta$. Note that any $\vec{D} \in \mathcal{C}_m(A)$ extending \vec{C}_n and such that $E_n \subseteq \bigcup \vec{D}$ will be a δ -proper extension of \vec{C}_n , hence we'll have $k(\vec{D}) < k(\vec{C}_n)$. Let $p(n+1)$ be the least integer $p \geq p(n)$ such that $E_i \in \mathcal{B}_p$, for all $i \leq n$. In order to define \mathcal{D}_{n+1} , Player I removes \vec{C}_n from \mathcal{D}_n and replaces it by all its δ -proper extensions in $\mathcal{C}_{m,p(n+1)}(A)$. If $k(\vec{C}_n) = 0$ there are no such extensions, so Player I simply removes \vec{C}_n from \mathcal{D}_n . Also,

observe that in the computation of α'_{n+1} , we replaced $\omega^{k(\vec{C}_n)}$ by finitely many ordinals of the form ω^l , for $l < k(\vec{C}_n)$. It follows that $\alpha'_{n+1} < \alpha'_n$. Since $\alpha_{n+1} = \alpha'_{n+1} + 1$, we also have that $\alpha_{n+1} < \alpha_n$ and, in addition, $\alpha_{n+1} \geq 1$. Clearly, Player I can play indefinitely by following this strategy. \square

Definition 5.5. Let α be an ordinal, m an integer, and $\delta > 0$. The game $\mathcal{H}(\alpha, m, \delta)$ is played between two players I and II as follows.

$$\begin{array}{ccccccc} \text{I :} & \alpha_0, B_0 & \alpha_1, B_1 & \cdots & \alpha_n, B_n & \cdots \\ \hline \text{II :} & E_0 & E_1 & \cdots & E_n & \cdots \end{array}$$

Player I plays a strictly decreasing sequence of ordinals $< \alpha$ and sets $B_n \in \mathcal{B}$ such that each B_n is (m, q_n) -thin, for some $q_n > m$. At stage n , Player II is required to play some $E_n \in \mathcal{B}$ that is disjoint from the E_i , for $i < n$, and such that $\nu(E_n \setminus B_n) \geq \delta$. The first player who cannot play following these rules loses.

Lemma 5.6. Suppose m is an integer and $\delta > 0$. Let $k = \lceil 4 \cdot |\mathcal{A}_m| / \delta \rceil$. Then Player I has a winning strategy in $\mathcal{H}(\omega^{k+2}, m, \delta)$.

Proof. Let $\delta' = \delta / |\mathcal{A}_m|$. By Lemma 5.4, we can fix a winning strategy σ_A for Player I in $\mathcal{G}(\omega^{k+1}, A, \delta')$, for all $A \in \mathcal{A}_m$. We describe a winning strategy σ for Player I in $\mathcal{H}(\omega^{k+2}, m, \delta)$. We think of playing all the games $\mathcal{G}(\omega^{k+1}, A, \delta')$ in parallel. In each of these games Player I follows his winning strategy σ_A . If Player II plays E_n in $\mathcal{H}(\omega^{k+2}, m, \delta)$ we consider that he plays $E_n \cap A$ in the game $\mathcal{G}(\omega^{k+1}, A, \delta')$. At stage n , let $(\alpha_n(A), C_n(A))$ be the n -th move of σ_A in the game $\mathcal{G}(\omega^{k+1}, A, \delta')$. For each $A \in \mathcal{A}_m$, let

$$H_n(A) = \begin{cases} \bigcup_{i < n} E_i \cap A, & \text{if } \nu(\bigcup_{i < n} E_i | A) \geq 2, \\ A \setminus C_n(A), & \text{otherwise.} \end{cases}$$

Let q_n be the least integer $q > m$ such that $H_n(A) \in \mathcal{B}_q$, for all $A \in \mathcal{A}_m$. Note that $\nu(H_n(A) | A) \geq 2$, for all $A \in \mathcal{A}_m$. Therefore, if we let $D_n(A) = A \setminus H_n(A)$, for all $A \in \mathcal{A}_m$, the set

$$B_n = \bigcup \{D_n(A) : A \in \mathcal{A}_m\},$$

is (m, q_n, ν) -thin. Let α_n be the natural sum of the $\alpha_n(A)$, for $A \in \mathcal{A}_m$. The strategy σ then plays (α_n, B_n) . Suppose that Player II responds by playing some E_n disjoint from the E_i , for $i < n$, and such that $\nu(E_n \setminus B_n) \geq \delta$. Then there must be some $A \in \mathcal{A}_m$ such that $\nu((E_n \setminus B_n) \cap A) \geq \delta'$. In particular, $\nu(E_n \cap H_n(A) | A) \geq \delta'$. If $H_n(A) = \bigcup_{i < n} E_i \cap A$, this is not possible since E_n is disjoint from the E_i , for $i < n$. Thus, it must be the case that $\nu(\bigcup_{i < n} E_i | A) < 2$ and $\nu(E_n \setminus C_n(A) | A) \geq \delta'$. This means that in the next move σ_A must play some pair $(\alpha_{n+1}(A), C_{n+1}(A))$, such that $\alpha_{n+1}(A) < \alpha_n(A)$. Since $\alpha_{n+1}(A') \leq \alpha_n(A')$, for all other $A' \in \mathcal{A}_m$, this means that $\alpha_{n+1} < \alpha_n$. Therefore, by doing this, Player I follows the rules in $\mathcal{H}(\omega^{k+2}, m, \delta)$. Finally, let us note that, for all $A \in \mathcal{A}_m$, the first move of σ_A is $(\omega^{k+1}, \emptyset)$. Hence, the first move of σ is $\omega^{k+1} \cdot |\mathcal{A}_m| < \omega^{k+2}$. Therefore σ is a winning strategy for Player I in $\mathcal{H}(\omega^{k+2}, m, \delta)$, as required. \square

We now introduced another game that will be used to bound the exhaustivity rank of our submeasure ν .

Definition 5.7. Let ξ be an ordinal. The game $\mathcal{E}(\xi)$ is played between two players I and II as follows.

$$\begin{array}{cccccc} \text{I :} & \xi_0 & \xi_1 & \cdots & \xi_n & \cdots \\ \hline \text{II :} & E_0 & E_1 & \cdots & E_n & \cdots \end{array}$$

Player I plays a strictly decreasing sequence of ordinals $\leq \xi$ and Player II plays pairwise disjoint sets $E_n \in \mathcal{B}$ such that $\nu(E_n) \geq 2^{-N} + \epsilon$. The first player who cannot play by following these rules loses.

Recall that we have assumed that $\|\cdot\|$ is an admissible norm and ν satisfies the N -thinness property relative to $\|\cdot\|$. We have defined $\mathcal{S} = \{F \in [\mathbb{N}]^{<\omega} : \|F\| < M_N\}$ and let $\beta = \rho(\mathcal{S})$.

Lemma 5.8. Player I has a winning strategy in the game $\mathcal{E}(\omega^{\omega \cdot (\beta+1)})$.

Proof. Let τ be a winning strategy for Player I in the game $\mathcal{G}_\beta(\mathcal{S})$ from Definition 2.4. For every m and $\delta > 0$, fix a winning strategy $\sigma_{m,\delta}$ for Player I in $\mathcal{H}(\omega^\omega, m, \delta)$. We combine those strategies into a winning strategy for Player I in $\mathcal{E}(\omega^{\omega \cdot (\beta+1)})$. Let us write ϵ_i for $\epsilon/2^{i+1}$. To avoid excessive notation, let us introduce some dynamic variables. First, l will denote an integer, F a set of integers of size $l+1$, and $\{m_0, \dots, m_l\}$ will denote the increasing enumeration of F . Also, $\vec{\gamma}$ will denote a decreasing sequence $(\gamma_0, \dots, \gamma_l)$ of ordinals $\leq \beta$ of length $l+1$. We will have that $(\gamma_0, m_0, \dots, \gamma_{l-1}, m_{l-1}, \gamma_l)$ is a position in $\mathcal{G}_\beta(\mathcal{S})$ in which Player I uses his strategy τ . In particular, we will have that $\{m_0, \dots, m_{l-1}\} \in \mathcal{S}$, but F itself may not be in \mathcal{S} . For each $i < l$ we will also fix a variable π_i denoting a certain position in the game $\mathcal{H}(\omega^\omega, m_i, \epsilon_i)$, in which Player I uses his winning strategy σ_{m_i, ϵ_i} and Player II plays some of the E_j from the game $\mathcal{E}(\omega^{\omega \cdot (\beta+1)})$. We denote the last move of Player I in π_i by (α_i, B_i) . We will also have that $B_i \in \mathcal{B}_{m_{i+1}}$ and is (m_i, m_{i+1}, ν) -thin. Given the value of all these variables at stage n we will compute a certain ordinal ξ_n which will be the move of Player I at that stage. Depending on the next move of Player II we will reset these variables for the next stage of the game.

To begin, set $l = 1$, $\gamma_0 = \beta$, $m_0 = 0$. Let γ_1 be the response of τ if Player II plays m_0 as his first move in the game $\mathcal{G}_\beta(\mathcal{S})$. Set $\vec{\gamma}$ to be (γ_0, γ_1) . Set π_0 to be the position in $\mathcal{H}(\omega^\omega, 0, \epsilon_0)$ after the first move of Player I given by the strategy σ_{0, ϵ_0} . Set m_1 to be the least integer q such that $B_0 \in \mathcal{B}_q$. Set F to be $\{m_0, m_1\}$.

Now, suppose we are at some stage n of the game $\mathcal{E}(\omega^{\omega \cdot (\beta+1)})$. Given the current values of the above variables, let s be such that the first move of σ_{m_l, ϵ_l} is $< \omega^s$. As his n -th move in $\mathcal{E}(\omega^{\omega \cdot (\beta+1)})$ Player I plays ξ_n equal to:

$$(1) \quad \omega^{\omega \cdot \gamma_0} \cdot \alpha_0 \oplus \omega^{\omega \cdot \gamma_1} \cdot \alpha_1 \oplus \dots \oplus \omega^{\omega \cdot \gamma_{l-1}} \cdot \alpha_{l-1} \oplus \omega^{\omega \cdot \gamma_l + s}.$$

Now, suppose Player II responds by playing some E_n disjoint from the E_i , for $i < n$, and such that $\nu(E_n) \geq 2^{-N} + \epsilon$. Let us describe how the above variables are reset. Consider the current values of the B_i , for $i < l$.

Case 1. Suppose first that $\nu(E_n \setminus B_i) < \epsilon_i$, for all i . Note that the set $B = \bigcap \{B_i : i < l\}$ is (F, ν) -thin. If $F \notin \mathcal{S}$ we have that $\|F\| = M_N$ and, hence, by the N -thinness property

of ν , we conclude that $\nu(B) \leq 2^{-N}$. But then we would have:

$$\nu(E_n) \leq \nu(B) + \nu(E_n \setminus B) \leq 2^{-N} + \sum_{i < l} \epsilon_i < 2^{-N} + \epsilon,$$

which is a contradiction. Now, if $F \in \mathcal{S}$ then m_l is a legitimate move for Player II in the position $(\gamma_0, m_0, \dots, m_{l-1}, \gamma_l)$ of $\mathcal{G}_\beta(\mathcal{S})$. We now reset the new value of l to be $l+1$. We set γ_{l+1} to be the move of τ in the position $(\gamma_0, m_0, \dots, \gamma_l, m_l)$ of the game $\mathcal{G}_\beta(\mathcal{S})$. We start a run π_l of $\mathcal{H}(\omega^\omega, m_l, \epsilon_l)$ by letting the strategy σ_{m_l, ϵ_l} make the first move, say (α_l, B_l) , in that game. We let m_{l+1} be the least integer $q \geq m_l$ such that $B_l \in \mathcal{B}_q$. We then add m_{l+1} to F . All other variables are kept unchanged. Let us consider the effect of these changes on (1). The first l terms have not changed. We have replaced $\omega^{\omega \cdot \gamma_l + s}$ by

$$\omega^{\omega \cdot \gamma_l} \cdot \alpha_l \oplus \omega^{\omega \cdot \gamma_{l+1} + s'}$$

for some integer s' . Note that $\alpha_l < \omega^s$ and $\gamma_{l+1} < \gamma_l$, hence the value of (1) decreases in the next stage of the game, i.e. $\xi_{n+1} < \xi_n$.

Case 2. Suppose now that $\nu(E_n \setminus B_i) \geq \epsilon_i$, for some i . Let j be the least such i . This means that E_n is a legitimate move for Player II in the current position π_j of $\mathcal{H}(\omega^\omega, m_j, \epsilon_j)$. We then let Player II play E_n in this position and we let σ_{m_j, ϵ_j} respond to this move. We set the resulting position to be our new π_j . We set the new value of l to be $j+1$. We keep all the positions π_i , for $i < j$, unchanged and we erase the positions π_i , for $i > j$. We keep the values of the γ_i , for $i \leq j+1$, unchanged and we erase the γ_i , for $i > j+1$. We keep all the m_i , for $i \leq j$, unchanged. For our new m_{j+1} we pick the least integer q such that the new B_j belongs to \mathcal{B}_q . We erase all the m_i , for $i > j+1$. Finally, we set $F = \{m_0, \dots, m_{j+1}\}$. In order to estimate the effect of these changes to (1) let us denote by α'_l the old value of α_l and by α''_l the new value of α_l . Let us also denote by α'_{l+1} the old value of α_{l+1} . The first $l-1$ terms of (1) have not changed. In the l -th term we replaced $\omega^{\omega \cdot \gamma_l} \cdot \alpha'_l$ by $\omega^{\omega \cdot \gamma_l} \cdot \alpha''_l$ and in the $l+1$ -th term we replaced $\omega^{\omega \cdot \gamma_{l+1}} \cdot \alpha'_{l+1}$ by $\omega^{\omega \cdot \gamma_{l+1} + s}$, for some integer s . We erased all later terms. Now, note that $\alpha''_l < \alpha'_l$ and $\gamma_{l+1} < \gamma_l$, hence,

$$\omega^{\omega \cdot \gamma_l} \cdot \alpha''_l \oplus \omega^{\omega \cdot \gamma_{l+1} + s} < \omega^{\omega \cdot \gamma_l} \cdot \alpha''_l + \omega^{\omega \cdot \gamma_l} = \omega^{\omega \cdot \gamma_l} \cdot (\alpha''_l + 1) \leq \omega^{\omega \cdot \gamma_l} \cdot \alpha'_l.$$

This means that the value of (1) decreases in the next stage of the game, i.e. $\xi_{n+1} < \xi_n$. Thus, Player I can continue playing in this way as long as Player II plays pairwise disjoint sets E_n with $\nu(E_n) \geq 2^{-N} + \epsilon$. This completes the proof of Lemma 5.8. \square

Now, combining Corollary 2.6, Lemma 2.10, Corollary 3.15 and Lemma 5.8, we obtain the following.

Corollary 5.9. *Suppose $0 < \alpha < \omega_1$. Let $\|\cdot\|_\alpha$ be the admissible norm derived from the α -th Schreier family and let ν^α be the associated exhaustive submeasure. Then*

$$\omega^\alpha \leq \text{rk}(\nu^\alpha) \leq \omega^{\omega \cdot (\alpha+1)^\omega}.$$

\square

REFERENCES

- [1] D. E. Alspach and S. Argyros. Complexity of weakly null sequences. *Dissertationes Math. (Rozprawy Mat.)*, 321:44, 1992.
- [2] I. Farah. Examples of ϵ -exhaustive pathological submeasures. *Fund. Math.*, 181(3):257–272, 2004.
- [3] V. Farmaki and S. Negrepontis. Block combinatorics. *Trans. Amer. Math. Soc.*, 358(6):2759–2779, 2006.
- [4] D. H. Fremlin. Talagrand’s example. preprint, available at: <https://www.essex.ac.uk/maths/people/fremlin/n06204.ps>, 2008.
- [5] D. H. Fremlin. *Measure theory. Vol. 5*. Torres Fremlin, Colchester, 2015.
- [6] D. Gale and F. M. Stewart. Infinite games with perfect information. In *Contributions to the theory of games, vol. 2*, Annals of Mathematics Studies, no. 28, pages 245–266. Princeton University Press, Princeton, N. J., 1953.
- [7] I. Gasparis and D. H. Leung. On the complemented subspaces of the Schreier spaces. *Studia Math.*, 141(3):273–300, 2000.
- [8] W. Hodges and S. Shelah. Infinite games and reduced products. *Ann. Math. Logic*, 20(1):77–108, 1981.
- [9] N. Kalton and J. Roberts. Uniformly exhaustive submeasures and nearly additive set functions. *Transactions of the American Mathematical Society*, 278:803–816, 1983.
- [10] A. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate texts in mathematics*. Springer, 1995.
- [11] D. Maharam. An algebraic characterization of measure algebras. *Annals of Mathematics*, 48:154–167, 1947.
- [12] R. D. Mauldin, editor. *The Scottish Book*. Birkhäuser Boston, Mass., 1981.
- [13] J. Roberts. Maharam’s problem. In P. Kranz and I. Labuda, editors, *Proceedings of the Orlitz memorial conference*. 1991. unpublished.
- [14] J. Roitman. A very thin thick superatomic Boolean algebra. *Algebra Universalis*, 21(2-3):137–142, 1985.
- [15] J. Schreier. Ein gegenbeispiel zur theorie der schwachen konvergenz. *Studia Mathematica*, 2(1):58–67, 1930.
- [16] W. Sierpiński. *Cardinal and ordinal numbers*. Second revised edition. Monografie Matematyczne, Vol. 34. Państwowe Wydawnictwo Naukowe, Warsaw, 1965.
- [17] M. Talagrand. Maharam’s problem. *Annals of Mathematics (2)*, 168(3):981–1009, 2008.
- [18] B. Veličković. Maharam algebras. *Ann. Pure Appl. Logic*, 158(3):190–202, 2009.

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